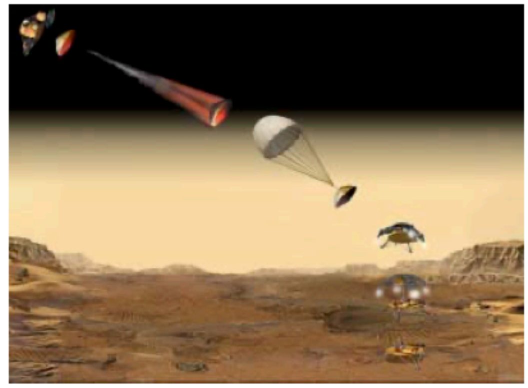
Chapter 2

Vectors ($\triangleq$ $=$ $-\vec{\mathbf{v}}$ $+$ $-$ $*$ $\cdot$ $\vec{\mathbf{v}}^2$ $\times$)

Examples in Hw 1, 2, 3

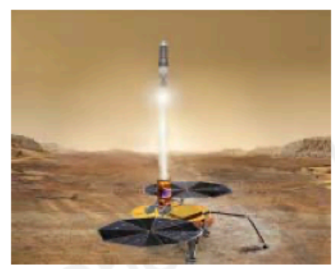


The word "***vector***" means different things in different contexts (as a complex number, part of a quaternion, row or column matrix, physical vector with magnitude and direction, etc.) and its complex multi-century history is tied to co-developments in mechanics and mathematics, with major contributions from dozens of famed scholars.

- Discovery (Cardan 1545) and geometrical representation (Caspar Wessel 1799, Gauss 1800$^+$) of complex numbers.
- Search for a geometry of position (Leibniz 1679). Idea of a parallelogram of forces or velocities (Newton 1687).
- Hamilton's 22-year development of quaternions (1843$^+$) via lectures, 109 papers, and 2 immense books. Hamilton (1846) introduces the terms "***scalar***" and "***vector***" in a quaternion $\hat{q} = \underbrace{q_0}_{\text{scalar}} + \underbrace{q_1\hat{i} + q_2\hat{j} + q_3\hat{k}}_{\text{vector}}$
- Gibbs (1846) uses "***vector***" to mean a quantity with magnitude and direction. The vector invented by Gibbs is sometimes called a physical, geometric, Gibbs, or Euclidean vector.
- Sylvester (1846) was the first to use the term "matrix" to describe an oblong arrangement of elements.

In 1881-1903, Gibbs developed ***vectors*** as a useful combination of magnitude and direction and invented their higher-dimensional counterparts ***dyadics***, ***triadics***, ***polyadics***. Vectors are an important **geometrical tool** (for surveying, motion, optics, graphics, CAD, Finite Element Analysis, ...).

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\hat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+$ $-$ $*$	Vector addition, negation, subtraction, and scalar multiplication/division.	Sections 2.6 - 2.8
$\cdot$ $\times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{^F d}{dt}$	Vector differentiation.	Chapters 7, 8



2.1 Examples of scalars vectors and dyadics

- A ***scalar*** is a number, possibly with units (e.g., 7 $\frac{\text{m}}{\text{s}}$ or 9 kg), such as

time	density	volume	mass	potential energy	work
distance	speed	angle	weight	kinetic energy	temperature

- A ***vector*** is a quantity with magnitude and ***one*** associated direction. For example, a ***velocity vector*** has speed (how fast something moves) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

force	velocity	acceleration	translational momentum
torque	angular velocity	angular acceleration	angular momentum

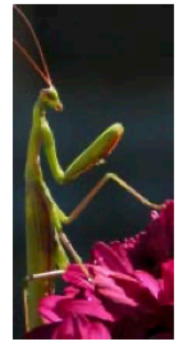
In 1884, Gibbs re-defined ***vector*** and taught vectors using **90** lectures.

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions. For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads**. For example, an ***inertia dyadic*** (Chapter 16) is the sum of dyads associated with moments and products of inertia.

- A ***triad*** is a quantity that has magnitude and ***three*** directions. A ***triadic*** is the sum of triads.

2.2 Definition of a vector

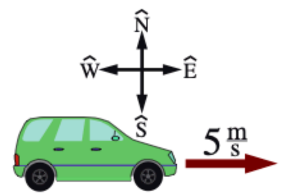
A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with an arrow and bold-faced font, e.g., $\vec{\mathbf{v}}$ denotes a vector.


Courtesy Bro. Claude Rheaume. LaSalette.

Certain vectors have additional properties, e.g., a ***position vector*** $\vec{\mathbf{r}}$ has two associated points and units of length (e.g., meters) and a ***unit vector*** has magnitude 1 (no units).

[a] A vector's ***magnitude*** is a real non-negative scalar (e.g., 7 m/s). A vector's ***direction*** is its ***orientation*** and ***sense***. A vector is similar to a **ray** in direction, but a vector has finite magnitude. A vector is similar to a **line segment** in magnitude and orientation, but a vector also has a ***sense*** (a fully defined direction).

Example of a vector: Consider the statement "the car is moving East at 5 $\frac{\text{m}}{\text{s}}$". It is convenient to represent the car's speed and direction with the velocity vector $\vec{\mathbf{v}} = 5\,\widehat{\mathbf{East}}$ (a hat designates the direction $\widehat{\mathbf{East}}$ as a ***unit vector***). The car's speed is always a real non-negative scalar denoted $|\vec{\mathbf{v}}|$ (the ***magnitude*** of $\vec{\mathbf{v}}$). The combination of ***magnitude*** and ***direction*** is a ***vector***.



The velocity of a car with speed 5 $\frac{\text{m}}{\text{s}}$ moving West can also be written as $\vec{\mathbf{v}} = \text{-}5\,\widehat{\mathbf{East}}$. The negative sign in $\text{-}5\,\widehat{\mathbf{East}}$ reverses vector $\vec{\mathbf{v}}$'s direction whereas $\vec{\mathbf{v}}$'s magnitude is $|\vec{\mathbf{v}}| = |\text{-}5\,\widehat{\mathbf{East}}| = 5\,\frac{\text{m}}{\text{s}}$.
Note: When $\vec{\mathbf{v}}$ is written as $\vec{\mathbf{v}} = \dot{x}\,\widehat{\mathbf{East}}$ where $\dot{x}$ is a scalar that can be **positive** or **zero** or **negative**, $\dot{x}$ is called the $\widehat{\mathbf{East}}$ ***measure*** of the vector $\vec{\mathbf{v}}$. The magnitude of $\vec{\mathbf{v}}$ is $|\vec{\mathbf{v}}| = \text{abs}(\dot{x})$ is inherently non-negative.

2.3 Zero vector $\vec{0}$, a vector whose magnitude is zero

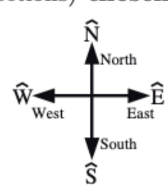
Addition with a zero vector:	$\mathbf{any}\vec{\mathbf{Vector}} + \vec{\mathbf{0}} = \mathbf{any}\vec{\mathbf{Vector}}$	
Dot product with a zero vector:	$\mathbf{any}\vec{\mathbf{Vector}} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors
Cross product with a zero vector:	$\mathbf{any}\vec{\mathbf{Vector}} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors
Derivative of the ***zero vector***:	$\frac{{}^F d\,\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and "anti-parallel" if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.[1]

2.4 Unit $\hat{\text{v}}$ectors: Vectors with magnitude 1 and no units (typeset with a $\widehat{\text{hat}}$)

Unit vectors are "***sign posts***" (e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ for local Earth directions) chosen to simplify communication and calculations. Other useful "sign posts" are:

- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector tangent to a curve or perpendicular to a surface

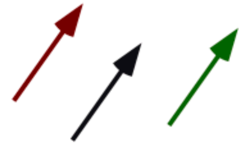


[1] The direction of a ***zero vector*** $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is ***parallel*** to all vectors, $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "the zero vector". It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$). To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\textbf{unit}\widehat{\mathbf{V}}\textbf{ector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \quad (1)$$

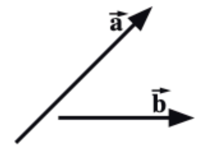
2.5 Equal vectors (=) vectors with the same magnitude and direction



Shown right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.

Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are ***equal force vectors*** when they have the same magnitude, direction, and point of application.

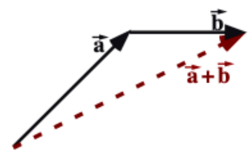
2.6 Vector addition (+)



As shown right, adding vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector. First $\vec{\mathbf{b}}$ is translated so its tail is at the tip of $\vec{\mathbf{a}}$. Next, $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $\vec{\mathbf{b}}$.

Translating $\vec{\mathbf{b}}$ does *not* change the magnitude or direction of $\vec{\mathbf{b}}$, and so produces an equal $\vec{\mathbf{b}}$.

Properties of vector addition



Commutative property: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$

Associative property: $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$

Addition of zero vector: $\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$

Vectors with different units do **not** add. Do **not** add a position vector (units of meters) with a force vector (units of Newtons).

Example: Vector addition (+) algebra

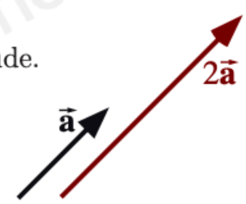
Shown right is how to add vectors $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$ that are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

$$\begin{array}{rcl} \vec{\mathbf{v}} &=& 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\ -\ \vec{\mathbf{w}} &=& 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \\ \hline \vec{\mathbf{v}} + \vec{\mathbf{w}} &=& 9\,\widehat{\mathbf{n}}_x + 8\,\widehat{\mathbf{n}}_y + 6\,\widehat{\mathbf{n}}_z \end{array}$$

$$\vec{\mathbf{v}} = \underbrace{x\,\widehat{\mathbf{n}}_x}_{\text{vector component}} + \underbrace{y\,\widehat{\mathbf{n}}_y}_{\text{vector component}}$$

Special names for parts of the generic vector $\vec{\mathbf{v}}$. x is called the $\widehat{\mathbf{n}}_x$ ***scalar component*** (***measure***) of $\vec{\mathbf{v}}$. y is called the $\widehat{\mathbf{n}}_y$ ***scalar component*** (***measure***) of $\vec{\mathbf{v}}$.

2.7 Vector multiplied or divided by a scalar (* or /)



- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{\mathbf{a}}$ by a scalar s is defined as $\frac{\vec{\mathbf{a}}}{s} \triangleq \frac{1}{s} * \vec{\mathbf{a}}$.

Properties of multiplication of a vector by a scalar s_1 or s_2

Commutative property: $s_1\,\vec{\mathbf{a}} = \vec{\mathbf{a}}\,s_1$

Associative property: $s_1\,(s_2\,\vec{\mathbf{a}}) = (s_1\,s_2)\,\vec{\mathbf{a}} = s_2\,(s_1\,\vec{\mathbf{a}}) = s_1\,s_2\,\vec{\mathbf{a}}$

Distributive property: $(s_1 + s_2)\,\vec{\mathbf{a}} = s_1\,\vec{\mathbf{a}} + s_2\,\vec{\mathbf{a}} \qquad s_1\,(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = s_1\,\vec{\mathbf{a}} + s_1\,\vec{\mathbf{b}}$

Multiplication by zero: $0 * \vec{\mathbf{a}} = \vec{\mathbf{0}}$

Example: Vector scalar multiplication and division (* and /)

Given: $\vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z$

then: $5\,\vec{\mathbf{v}} = 35\,\widehat{\mathbf{n}}_x + 25\,\widehat{\mathbf{n}}_y + 20\,\widehat{\mathbf{n}}_z$ and $\frac{\vec{\mathbf{v}}}{-2} = -3.5\,\widehat{\mathbf{n}}_x - 2.5\,\widehat{\mathbf{n}}_y - 2\,\widehat{\mathbf{n}}_z$

2.8 Vector negation and subtraction $(-)$

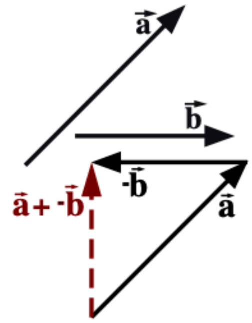
Negation: As shown right, negating a vector (multiplying by -1) reverses the vector's ***sense*** (it points in the opposite direction). Negation does not change the vector's magnitude or orientation.

Subtraction: As the drawing to the right shows, subtracting a vector $\vec{b}$ from a vector $\vec{a}$ is simply addition and negation.[a]

$$\vec{a} - \vec{b} \triangleq \vec{a} + \text{-}\vec{b}$$

[a] In most/all mathematics, subtraction is defined as negation and addition.

After negating vector $\vec{b}$, it is translated so the tail of -$\vec{b}$ is at the tip of $\vec{a}$.
Next, vector $\vec{a}$ + -$\vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated -$\vec{b}$.



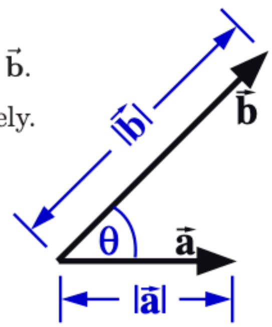
Example: Vector subtraction $(\vec{v} - \vec{w})$
It is easy to subtract vectors that are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.

$$\begin{aligned} \vec{v} &= 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\ -\ \vec{w} &= 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \\ \hline \vec{v} - \vec{w} &= 5\,\widehat{\mathbf{n}}_x + 2\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \end{aligned}$$

2.9 Vector dot product $(\cdot)$

Equation (2) defines the ***dot product*** of vectors $\vec{a}$ and $\vec{b}$.

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively.
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.



$$\vec{a} \cdot \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\cos(\theta) \qquad (2)$$

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \qquad (3)$$

Use **acos** to calculate θ.

Note: $\vec{a}$ and $\vec{b}$ are "***perpendicular***" when $\vec{a} \cdot \vec{b} = 0$.
Note: Dot-products encapsulate the ***law of cosines***.

Equation (2) shows $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{v}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{v}^{\,n}$ (vector $\vec{v}$ raised to scalar power n) as a non-negative scalar.

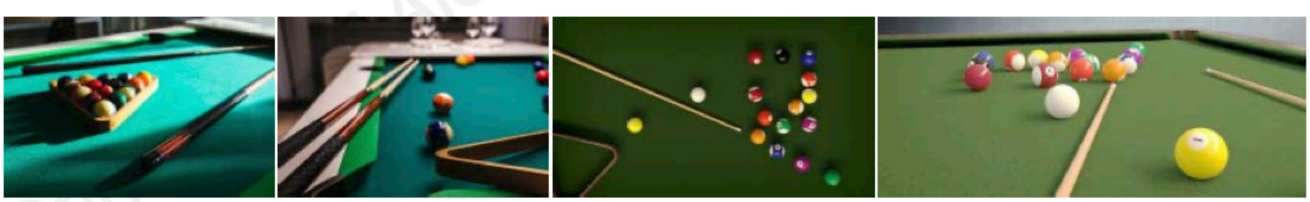
$$\begin{aligned} \vec{v}^{\,2} \triangleq |\vec{v}|^2 &= \vec{v} \cdot \vec{v} \\ |\vec{v}| &= +\sqrt{\vec{v} \cdot \vec{v}} \\ \vec{v}^{\,n} \triangleq |\vec{v}|^n &= +(\vec{v} \cdot \vec{v})^{\frac{n}{2}} \end{aligned} \qquad (4)$$

Example: Kinetic energy $K = \frac{1}{2}\,m\,\vec{v}^{\,2} \underset{(4)}{=} \frac{1}{2}\,m\,\vec{v} \cdot \vec{v}$

2.9.1 Properties of the dot-product $(\cdot)$

Dot product with a zero vector	$\vec{a} \cdot \vec{0} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{a} \cdot \vec{b} = 0$	if $\vec{a} \perp \vec{b}$
Dot product of parallel vectors	$\vec{a} \cdot \vec{b} = \pm\,|\vec{a}|\,|\vec{b}|$	if $\vec{a} \parallel \vec{b}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{a} \cdot s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \cdot \vec{b})$	
Commutative property	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	
Distributive property	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$	
Distributive property	$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$	

Note: The distributive property for dot-products and cross-products is proved in [37, pgs. 23-24, 32-34].



2.9.2 Uses for the dot-product $(\cdot)$

- Calculating an ***angle*** between two vectors [see equation (3) and example in Section 3.4]
- Determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.1 and 3.4].
- Changing a ***vector equation*** into a ***scalar equation*** (see Hw 2.33).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [from equation (1)] $\quad \mathbf{unit}\vec{\mathbf{V}}\mathbf{ector} \underset{(1)}{=} \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$, defined as: $\quad \vec{\mathbf{v}} \cdot \dfrac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$
  See Section 4.6 for ***projections***, ***measures***, ***coefficients***, ***components***.
  See Section 3.4 for a distance measure from a point to a plane.

  Projection of $\vec{\mathbf{v}}$ onto the plane N perpendicular to $\widehat{\mathbf{n}}$: $\text{Proj}(\vec{\mathbf{v}})_{\text{N}} = \vec{\mathbf{v}} - (\vec{\mathbf{v}} \cdot \widehat{\mathbf{n}})\,\widehat{\mathbf{n}} = \widehat{\mathbf{n}} \times (\vec{\mathbf{v}} \times \widehat{\mathbf{n}})$.
  Context: $\vec{\mathbf{v}}$ is a vector "bound" to a point v_o whose position vector $\vec{\mathbf{r}}$ from a point N_o fixed in N has $\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}} > 0$.
  Example: Projection of a position vector $\vec{\mathbf{r}}$ (from N_o to a point R, where $\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}} > 0$) onto N: $\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}})\,\widehat{\mathbf{n}}$.

  Projection of a parallelogram characterized by $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ onto plane N: $\big|\text{Proj}(\vec{\mathbf{p}})_{\text{N}} \times \text{Proj}(\vec{\mathbf{q}})_{\text{N}}\big|\,\widehat{\mathbf{n}}$.
  Magnitude of $\vec{\mathbf{p}}$'s projection on N crossed-with $\vec{\mathbf{q}}$'s projection on N times $\widehat{\mathbf{n}}$: $\big|\vec{\mathbf{p}} \times \vec{\mathbf{q}} + [(\vec{\mathbf{p}} \cdot \widehat{\mathbf{n}})\,\vec{\mathbf{q}} - (\vec{\mathbf{q}} \cdot \widehat{\mathbf{n}})\,\vec{\mathbf{p}}] \times \widehat{\mathbf{n}}\big|\,\widehat{\mathbf{n}}$.

2.9.3 Dot-products to change vector equations to scalar equations (see Hw 1.33)

One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, i.e.,

Method 1: $\quad$ if $\quad \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_3 = 0$

Section 2.11.2 describes another way to form three ***different*** scalar equations from $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.9.4 Special case: Dot-products with orthogonal unit vectors (and matrix multiplication)

When $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$ are **orthogonal unit** vectors, it can be shown (see Hw 2.4)

$$(a_x\,\widehat{\mathbf{n}}_{\text{x}} + a_y\,\widehat{\mathbf{n}}_{\text{y}} + a_z\,\widehat{\mathbf{n}}_{\text{z}}) \cdot (b_x\,\widehat{\mathbf{n}}_{\text{x}} + b_y\,\widehat{\mathbf{n}}_{\text{y}} + b_z\,\widehat{\mathbf{n}}_{\text{z}}) \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

Optional: This special case dot-product happens to be equal to the multiplication of the $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$ row matrix representation of the first vector with the $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$ column matrix representation of the second vector as

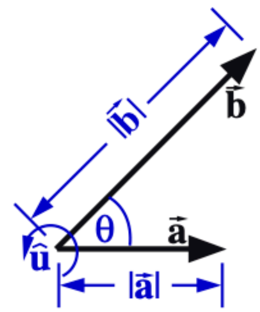
$$\textbf{Matrix multiplication:} \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \;=\; \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}_{\widehat{\mathbf{n}}_{\text{xyz}}} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_{\widehat{\mathbf{n}}_{\text{xyz}}} \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

2.9.5 Examples: Vector dot-products $(\cdot)$

Shown below is how to use dot-products when vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$. $\quad \vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_{\text{x}} + 5\,\widehat{\mathbf{n}}_{\text{y}} + 4\,\widehat{\mathbf{n}}_{\text{z}}$, $\quad \vec{\mathbf{w}} = 2\,\widehat{\mathbf{n}}_{\text{x}} + 3\,\widehat{\mathbf{n}}_{\text{y}} + 2\,\widehat{\mathbf{n}}_{\text{z}}$

$\widehat{\mathbf{n}}_{\text{x}}$ measure of $\vec{\mathbf{v}}$	$\vec{\mathbf{v}} \cdot \widehat{\mathbf{n}}_{\text{x}} = 7$ (**measures** how much of $\vec{\mathbf{v}}$ is in the $\widehat{\mathbf{n}}_{\text{x}}$ direction).
$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 7^2 + 5^2 + 4^2 = 90$	$\lvert\vec{\mathbf{v}}\rvert = \sqrt{90} \approx 9.4868$
$\vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = 2^2 + 3^2 + 2^2 = 17$	$\lvert\vec{\mathbf{w}}\rvert = \sqrt{17} \approx 4.1231$
Unit vector in the direction of $\vec{\mathbf{v}}$:	$\dfrac{\vec{\mathbf{v}}}{\lvert\vec{\mathbf{v}}\rvert} = \dfrac{7\,\widehat{\mathbf{n}}_{\text{x}} + 5\,\widehat{\mathbf{n}}_{\text{y}} + 4\,\widehat{\mathbf{n}}_{\text{z}}}{\sqrt{90}} \approx 0.738\,\widehat{\mathbf{n}}_{\text{x}} + 0.527\,\widehat{\mathbf{n}}_{\text{y}} + 0.422\,\widehat{\mathbf{n}}_{\text{z}}$
Unit vector in the direction of $\vec{\mathbf{w}}$:	$\dfrac{\vec{\mathbf{w}}}{\lvert\vec{\mathbf{w}}\rvert} = \dfrac{2\,\widehat{\mathbf{n}}_{\text{x}} + 3\,\widehat{\mathbf{n}}_{\text{y}} + 2\,\widehat{\mathbf{n}}_{\text{z}}}{\sqrt{17}} \approx 0.485\,\widehat{\mathbf{n}}_{\text{x}} + 0.728\,\widehat{\mathbf{n}}_{\text{y}} + 0.485\,\widehat{\mathbf{n}}_{\text{z}}$
$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 7*2 + 5*3 + 4*2 = 37$	$\angle(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \text{acos}\big(\frac{37}{\sqrt{90}\,\sqrt{17}}\big) \approx 0.33 \text{ rad} \approx 18.93°$

2.10 Vector cross product $(\times)$

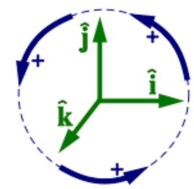


The ***cross product*** of a vector $\vec{a}$ with a vector $\vec{b}$ is defined in equation (5).

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.
- $\hat{u}$ is the unit vector **perpendicular** to both $\vec{a}$ and $\vec{b}$.
  The direction of $\hat{u}$ is determined by the ***right-hand rule***.
  The right-hand rule is a convention like driving on the right-hand side of the road. Until 1965, the Soviet Union used the left-hand rule.
  Note: $|\vec{a}|\,|\vec{b}|\,\sin(\theta)$ [the coefficient of $\hat{u}$ in equation (5)] is inherently non-negative because $\sin(\theta) \ge 0$ since $0 \le \theta \le \pi$. Hence, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin(\theta)$.

$$\boxed{\vec{a} \times \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\sin(\theta)\,\hat{u}} \qquad (5)$$

Properties of the cross-product $(\times)$



Cross product with a zero vector	$\vec{a} \times \vec{0} = \vec{0}$
Cross product of a vector with itself	$\vec{a} \times \vec{a} = \vec{0}$
Cross product of ***parallel*** vectors	$\vec{a} \times \vec{b} = \vec{0} \qquad \text{if } \vec{a} \parallel \vec{b}$
Cross product of scaled vectors	$s_1\,\vec{a} \times s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \times \vec{b})$
Distributive property	$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
Cross products are ***not*** associative	$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$
Remove $\hat{u}$ component of vector $\vec{v}$	$\hat{u} \times (\vec{v} \times \hat{u}) \underset{(7)}{=} \vec{v}\,(\overset{1}{\hat{u} \cdot \hat{u}}) - \hat{u}\,(\vec{v} \cdot \hat{u}) = \vec{v} - \hat{u}\,(\vec{v} \cdot \hat{u})$
Form $\vec{v}$ from $\vec{v} \cdot \hat{u}$ and $\vec{v} \times \hat{u}$	$\vec{v} = (\vec{v} \cdot \hat{u})\,\hat{u} + \hat{u} \times (\vec{v} \times \hat{u})$ ($\hat{u}$ is an arbitrary unit vector)
For any unit vector $\hat{u}$	$\lvert\vec{a} \times \hat{u}\rvert^2 \underset{(13)}{=} \vec{a} \cdot \vec{a} - (\vec{a} \cdot \hat{u})^2$ (proved in Hw 2.23).

Cross products are ***not*** commutative.

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}} \qquad (6)$$

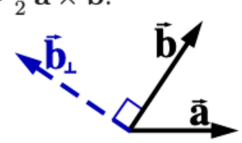
Vector triple cross product (bac-cab).

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})} \qquad (7)$$

A mnemonic for eqn (7) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$ is "back cab" - as in were you born in the back of a cab? Many proofs of this formula resolve $\vec{a}$, $\vec{b}$, and $\vec{c}$ into orthogonal unit vectors (e.g., $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$) and equate components.

2.10.1 Uses for the cross-product $(\times)$ in geometry, statics, motion analysis, ...

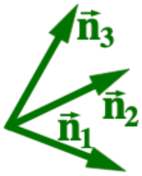
- ***Moment*** of a force such as $\vec{r} \times \vec{F}$ (details in Section 19.1).
- ***Velocity/acceleration*** formulas [see eqns (10.3, 10.4)] $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$.
- ***Perpendicular*** vectors, e.g., $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
- ***Distance*** between a point and a line (see Section 3.2 and example in Section 3.4).
- ***Area of a triangle*** with sides $\vec{a}$ and $\vec{b}$ (see Sections 3.3, 3.4 and Hw 2.17). $\vec{\Delta}(\vec{a}, \vec{b}) = \frac{1}{2}\,\vec{a} \times \vec{b}$.
- The vector $\vec{b}_\perp$ (shown right) is perpendicular to $\vec{b}$ and is in the plane containing both $\vec{a}$ and $\vec{b}$. It is calculated with the ***vector triple cross product*** as $\vec{b}_\perp = (\vec{a} \times \vec{b}) \times \vec{b}$. In general, $|\vec{b}_\perp| \neq |\vec{b}|$ and $\vec{b}_\perp$ is not perpendicular to $\vec{a}$.



2.10.2 Determinants and cross-products (with right-handed unit vectors)

When vectors $\vec{a}$ and $\vec{b}$ are expressed in terms of **orthogonal unit** vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, it can be shown (Hw 2.12) that $\vec{a} \times \vec{b}$ happens to equal the ***determinant*** of an associated matrix.

$$\left.\begin{aligned} \vec{a} &= a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k} \\ \vec{b} &= b_x\,\hat{i} + b_y\,\hat{j} + b_z\,\hat{k} \end{aligned}\right\} \qquad \vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{aligned} &(a_y\,b_z - a_z\,b_y)\,\hat{i} \\ -\;&(a_x\,b_z - a_z\,b_x)\,\hat{j} \\ +\;&(a_x\,b_y - a_y\,b_x)\,\hat{k} \end{aligned} \qquad (8)$$



Similarly for the cross product $\vec{c} \times \vec{d}$ with **non-orthogonal non-unit** vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$.

$$\left.\begin{aligned}\vec{c} &= c_1\,\vec{n}_1 + c_2\,\vec{n}_2 + c_3\,\vec{n}_3 \\ \vec{d} &= d_1\,\vec{n}_1 + d_2\,\vec{n}_2 + d_3\,\vec{n}_3\end{aligned}\right\} \quad \vec{c} \times \vec{d} = \det\begin{bmatrix}\vec{n}_2 \times \vec{n}_3 & \vec{n}_3 \times \vec{n}_1 & \vec{n}_1 \times \vec{n}_2 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3\end{bmatrix} = \begin{array}{r}(c_2\,d_3 - c_3\,d_2)\;\vec{n}_2 \times \vec{n}_3 \\ -\;(c_1\,d_3 - c_3\,d_1)\;\vec{n}_3 \times \vec{n}_1 \\ +\;(c_1\,d_2 - c_2\,d_1)\;\vec{n}_1 \times \vec{n}_2\end{array}$$

Examples: Vector cross-products ($\times$) with determinants.

The following shows how to use cross-products with the vectors $\vec{v}$ and $\vec{w}$, each which is expressed in terms of the orthogonal unit vectors $\widehat{n}_x$, $\widehat{n}_y$, $\widehat{n}_z$ shown to the right.

$$\left.\begin{aligned}\vec{v} &= 7\hat{i} + 5\hat{j} + 4\hat{k} \\ \vec{w} &= 2\hat{i} + 3\hat{j} + 2\hat{k}\end{aligned}\right\} \qquad \vec{v} \times \vec{w} = \det\begin{bmatrix}\hat{i} & \hat{j} & \hat{k} \\ 7 & 5 & 4 \\ 2 & 3 & 2\end{bmatrix} = \text{-}2\hat{i} - 6\hat{j} + 11\hat{k}$$

$$\text{Scalar triple product:} \quad (2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (\vec{v} \times \vec{w}) = \det\begin{bmatrix}2 & 3 & 4 \\ 7 & 5 & 4 \\ 2 & 3 & 2\end{bmatrix} = 22$$

2.10.3 Optional: Skew-symmetric matrices, cross products, and dot products

As shown in eqns (9) and (11), a cross-product can be performed with ***skew symmetric matrix multiplication***. Albeit ***inefficient*** for calculations (see Hw 2.13), it is useful in ***proofs*** (e.g., see Section 9.6.9). Eqns (10) and (12), show these skew-symmetric matrices in terms of dot-products.

Orthogonal unit vectors

$$\left.\begin{aligned}\vec{a} &= a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k} \\ \vec{b} &= b_x\,\hat{i} + b_y\,\hat{j} + b_z\,\hat{k}\end{aligned}\right\} \qquad \vec{a} \times \vec{b} = \begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\end{bmatrix} \underbrace{\begin{bmatrix}0 & \text{-}a_z & a_y \\ a_z & 0 & \text{-}a_x \\ \text{-}a_y & a_x & 0\end{bmatrix}}_{\text{skew}[\vec{a}]_{ijk}} \begin{bmatrix}b_x \\ b_y \\ b_z\end{bmatrix} \qquad (9)$$

$$\text{skew}[\vec{a}]_{ijk} = \begin{bmatrix}0 & \text{-}a_z & a_y \\ a_z & 0 & \text{-}a_x \\ \text{-}a_y & a_x & 0\end{bmatrix} = \text{-}\vec{a} \cdot \begin{bmatrix}\vec{0} & \hat{k} & \text{-}\hat{j} \\ \text{-}\hat{k} & \vec{0} & \hat{i} \\ \hat{j} & \text{-}\hat{i} & \vec{0}\end{bmatrix} = \text{-}\vec{a} \cdot \begin{bmatrix}\hat{i}\times\hat{i} & \hat{i}\times\hat{j} & \hat{i}\times\hat{k} \\ \hat{j}\times\hat{i} & \hat{j}\times\hat{j} & \hat{j}\times\hat{k} \\ \hat{k}\times\hat{i} & \hat{k}\times\hat{j} & \hat{k}\times\hat{k}\end{bmatrix} = \text{-}\vec{a} \cdot \begin{bmatrix}\hat{i} \\ \hat{j} \\ \hat{k}\end{bmatrix} \times \begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\end{bmatrix} \qquad (10)$$

Non-orthogonal non-unit vectors

$$\left.\begin{aligned}\vec{c} &= c_1\,\vec{n}_1 + c_2\,\vec{n}_2 + c_3\,\vec{n}_3 \\ \vec{d} &= d_1\,\vec{n}_1 + d_2\,\vec{n}_2 + d_3\,\vec{n}_3\end{aligned}\right\} \qquad \vec{c} \times \vec{d} = \begin{bmatrix}\vec{n}_2 \times \vec{n}_3 & \vec{n}_3 \times \vec{n}_1 & \vec{n}_1 \times \vec{n}_2\end{bmatrix} \underbrace{\begin{bmatrix}0 & \text{-}c_3 & c_2 \\ c_3 & 0 & \text{-}c_1 \\ \text{-}c_2 & c_1 & 0\end{bmatrix}}_{\text{skew}[\vec{c}]_n} \begin{bmatrix}d_1 \\ d_2 \\ d_3\end{bmatrix} \qquad (11)$$

$$\text{skew}[\vec{c}]_n = \begin{bmatrix}0 & \text{-}c_3 & c_2 \\ c_3 & 0 & \text{-}c_1 \\ \text{-}c_2 & c_1 & 0\end{bmatrix} \underset{(4.2)}{=} \frac{\text{-}\vec{c}}{\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)} \cdot \begin{bmatrix}\vec{n}_1 \times \vec{n}_1 & \vec{n}_1 \times \vec{n}_2 & \vec{n}_1 \times \vec{n}_3 \\ \vec{n}_2 \times \vec{n}_1 & \vec{n}_2 \times \vec{n}_2 & \vec{n}_2 \times \vec{n}_3 \\ \vec{n}_3 \times \vec{n}_1 & \vec{n}_3 \times \vec{n}_2 & \vec{n}_3 \times \vec{n}_3\end{bmatrix} = \frac{\text{-}\vec{c}}{\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)} \cdot \begin{bmatrix}\vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3\end{bmatrix} \times \begin{bmatrix}\vec{n}_1 & \vec{n}_2 & \vec{n}_3\end{bmatrix} \qquad (12)$$

Note: In general, a skew-symmetric matrix S is a square matrix with the property $S^T = \text{-}S$ (its transpose is its negative), so S has zeros along its diagonal. Eqn (5.10) relates skew-symmetric matrices expressed in bases a and b as $\text{skew}[\vec{v}]_b = {}^bR^a\;\text{skew}[\vec{v}]_a\;{}^aR^b$. Section 15.7 discusses skew-symmetric matrices and cross-products with dyadics and the unit dyadic.

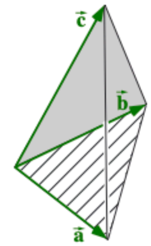
2.11 Scalar triple product $(\cdot \ \times \ \text{or} \ \times \ \cdot)$

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ is the scalar defined in the various ways shown below. Hw 2.15 shows how ***determinants*** can calculate scalar triple products.

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \ = \ \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} \ = \ \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} \ = \ \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \qquad (13)$$

Although parentheses help clarify equation (13) e.g., $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ instead of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product (for a sensible result to be produced).

2.11.1 Scalar triple product and the volume of a tetrahedron

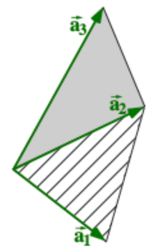


For a tetrahedron whose sides are described by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ (sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, $|\vec{\mathbf{c}}|$), a geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of the parallelepiped***. This formula

helps calculate mass and volume of generic 3D shapes (e.g., for highway cut/fill calculations and CAD/CAE solid modeling). A tetrahedron's volume is calculated in Section 3.4.

$$\boxed{\text{Tetrahedron Volume} \ = \ \tfrac{1}{6}\,\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \ = \ \tfrac{1}{6}\,\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \ \underset{(3.4)}{=} \ \tfrac{1}{3}\,\vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}} \qquad (14)$$

2.11.2 $(\times \ \cdot)$ to change vector equations to scalar equations (see Hw 1.33)



Section 2.9.3 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below [proved by directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into eqn (4.2)].

Method 2: if $\quad \vec{\mathbf{v}} \ = \ v_1\,\vec{\mathbf{a}}_1 \ + \ v_2\,\vec{\mathbf{a}}_2 \ + \ v_3\,\vec{\mathbf{a}}_3 \ = \ \vec{\mathbf{0}} \quad \Rightarrow \quad \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$

2.12 Words: Vectors vs. 1D matrices in the context of $\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}$

Language is complicated and words require context. For example, some words are contranyms (have opposite meanings) such as "fast" and "bolt" (move quickly or fasten), "buckle" (connect or collapse), or "dust" (remove or add particles). The word ***vector*** has **different** meanings in physics and matrix algebra. To confuse matters further, **sometimes** these two different types of vectors can be interrelated.

Vector (Gibbs/geometrical/physical vector)	Quantity with magnitude and direction, helpful for $\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}$
[Vector] as a row or column matrix	One-dimensional (1D) array or matrix, **without** direction, e.g., $[\text{"Sue"} \ \ \text{"Amy"} \ \ \text{"Bob"}]$ and $[2 \ \ 3 \ \ \sqrt{-4}]$ do not have direction.

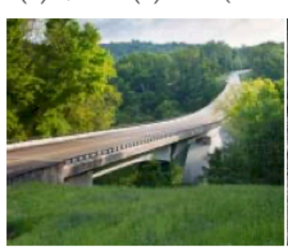
As shown below, it is possible to make a 1D matrix into a Gibbs/geometrical vector by attaching a basis.

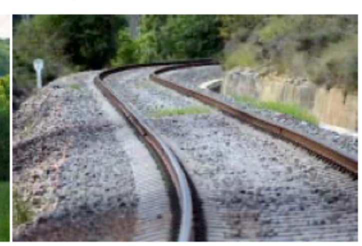
$$\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{a}}_x \\ \hat{\mathbf{a}}_y \\ \hat{\mathbf{a}}_z \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\hat{\mathbf{a}}_{xyz}}$$

where $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are orthogonal unit vectors.

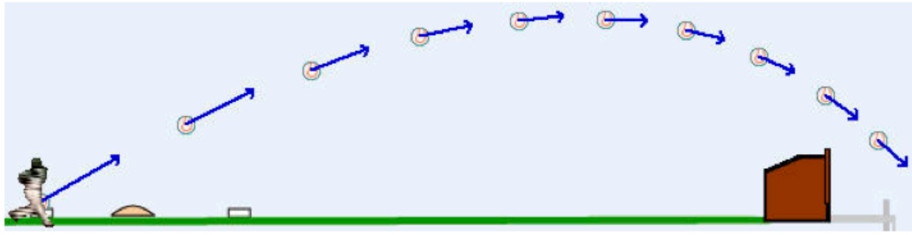
Related: Hw 2.3.

Note: Sometimes, representing a vector with a row or column matrix and orthogonal unit vectors can be inefficient. At times, it can be helpful to postpone resolving a vector into orthogonal components to allow maximum use of simplifying vector properties and avoid unnecessary simplifications such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Hw 2.10).





Courtesy Accuray Inc. Vectors are widely useful, e.g., in medical robotics, cut/fill calculations for highway & railway construction, ...

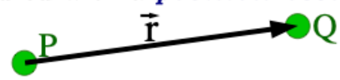


Chapter 3

Position vectors and vector geometry

3.1 Position of a point (or particle) (see examples in Hw 3)

A ***point*** is a location in space with no spatial dimension (no height, width, or depth). A ***particle*** is a point with mass (all particles are points, but not all points are particles). ***center of mass*** is a ***point*** that plays a central role for gravity and $\vec{\mathbf{F}} = \text{m}\,\vec{\mathbf{a}}$. A point's location can be measured with a ***position vector*** that characterizes its position from another point.

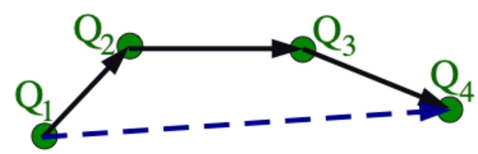


A position vector is defined by its properties: two points associated with a vector having units of length. For points P and Q, the symbol ${}^{P}\vec{\mathbf{r}}^{Q}$ denotes the position from P to Q.
The magnitude $|{}^{P}\vec{\mathbf{r}}^{Q}|$ is the ***distance*** between P and Q.

Distance between two points

$$|{}^{P}\vec{\mathbf{r}}^{Q}| = +\sqrt{{}^{P}\vec{\mathbf{r}}^{Q} \cdot {}^{P}\vec{\mathbf{r}}^{Q}} \tag{1}$$

Usually, a position vector is formed by **inspection** or **vector addition**. For example, shown right are points Q_1, Q_2, Q_3, Q_4. The position vector ${}^{Q_1}\vec{\mathbf{r}}^{Q_4}$ (from Q_1 to Q_4) is formed by vector addition as shown in equation (2).

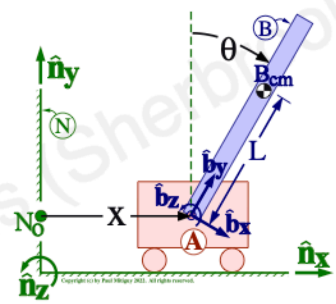


$${}^{Q_1}\vec{\mathbf{r}}^{Q_4} = {}^{Q_1}\vec{\mathbf{r}}^{Q_2} + {}^{Q_2}\vec{\mathbf{r}}^{Q_3} + {}^{Q_3}\vec{\mathbf{r}}^{Q_4} \tag{2}$$

Squash rule for adding position vectors

Position vectors are **very useful for geometry**, e.g., for ***angles*** [Hw 2.4 4.22], ***distance*** [Hw 2.10], ***area*** [Hw 2.17], and ***location*** [Hw 2.19, 4.23, 7.1].

Example: Position vector and distance (inverted pendulum on cart)



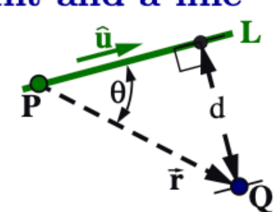
The ***position vector*** position from N_o to B_{cm} from point N_o to point B_{cm} is determined by visual inspection (from the figure to the right) and vector addition.

- Visual inspection: Position from N_o to A: ${}^{N_o}\vec{\mathbf{r}}^{A} = x\,\widehat{\mathbf{n}}_{\text{x}}$ Position from A to B_{cm}: ${}^{A}\vec{\mathbf{r}}^{B_{cm}} = L\,\widehat{\mathbf{b}}_{\text{y}}$
- Vector addition: ${}^{N_o}\vec{\mathbf{r}}^{B_{cm}} = {}^{N_o}\vec{\mathbf{r}}^{A} + {}^{A}\vec{\mathbf{r}}^{B_{cm}} = x\,\widehat{\mathbf{n}}_{\text{x}} + L\,\widehat{\mathbf{b}}_{\text{y}}$
- The ***distance*** d between N_o and B_{cm} is $|{}^{N_o}\vec{\mathbf{r}}^{B_{cm}}|$ (the magnitude of ${}^{N_o}\vec{\mathbf{r}}^{B_{cm}}$).

$$\text{d} \underset{(1)}{=} +\sqrt{{}^{N_o}\vec{\mathbf{r}}^{B_{cm}} \cdot {}^{N_o}\vec{\mathbf{r}}^{B_{cm}}} = +\sqrt{(x\,\widehat{\mathbf{n}}_{\text{x}} + L\,\widehat{\mathbf{b}}_{\text{y}}) \cdot (x\,\widehat{\mathbf{n}}_{\text{x}} + L\,\widehat{\mathbf{b}}_{\text{y}})} = +\sqrt{x^2 + 2\,x\,L\,\sin(\theta) + L^2}$$

3.2 Distance between a point and a line

A line L passes through a point P and is parallel to the unit vector $\widehat{\mathbf{u}}$. The distance d between line L and a point Q can be calculated as shown right. Other distance calculations are in Sections 3.4, 3.6 and Hw 1.26.



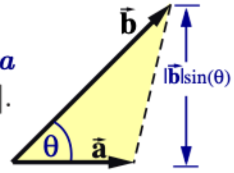
Distance between point Q and line L

$$\text{d} = |\vec{\mathbf{r}} \times \widehat{\mathbf{u}}| \underset{(2.5)}{=} |\vec{\mathbf{r}}|\,\sin(\theta) \tag{3}$$

where $\vec{\mathbf{r}} = {}^{P}\vec{\mathbf{r}}^{Q}$

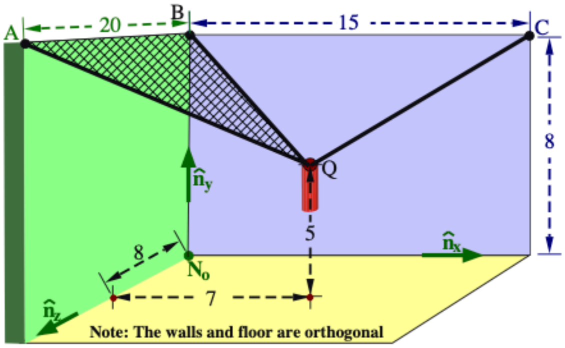
3.3 Area of a triangle

Eqn (4) calculates the vector or scalar ***area of a triangle*** with sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$.
Surveying: Hw 2.17 and Section 3.4 show how to use cross-products to calculate area.



$$\begin{aligned} \text{Vector area} \quad \vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) &= \tfrac{1}{2}\, \vec{\mathbf{a}} \times \vec{\mathbf{b}} \\ \text{Scalar area} \quad \Delta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) &= \tfrac{1}{2}\, |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| \\ \tfrac{1}{2}\, \text{base} * \text{height} &= \tfrac{1}{2}\, |\vec{\mathbf{a}}|\, |\vec{\mathbf{b}}|\, \sin(\theta) \end{aligned} \qquad (4)$$

3.4 Geometry example: Length/distances, angle, surface area, volume



Three cables attach a microphone Q to pegs A, B, C.
Given: Peg and microphone locations from a point N_o.

Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q's height above N_o	5 m
Distance along side wall (see picture)	8 m

$$^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$$

Length L_A of the cable joining A and Q.

- Form the position from N_o to A (inspection). $\quad {}^{N_o}\vec{\mathbf{r}}^{\,A} = 8\,\hat{\mathbf{n}}_y + 20\,\hat{\mathbf{n}}_z$
- Form the position from A to Q (vector addition). $\quad {}^{A}\vec{\mathbf{r}}^{\,Q} = {}^{A}\vec{\mathbf{r}}^{\,N_o} + {}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z$
- Calculate ${}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,Q}$. $\quad (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) \cdot (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) = 202$
- Form $|{}^{A}\vec{\mathbf{r}}^{\,Q}|$, the magnitude of ${}^{A}\vec{\mathbf{r}}^{\,Q}$. $\quad L_A = |{}^{A}\vec{\mathbf{r}}^{\,Q}| = \sqrt{{}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,Q}} = \sqrt{202} \approx 14.2$

Angle ϕ between lines $\overline{AQ}$ and $\overline{AB}$.

- To solve for ϕ, rearrange the dot-product: $\quad {}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,B} \triangleq |{}^{A}\vec{\mathbf{r}}^{\,Q}|\, |{}^{A}\vec{\mathbf{r}}^{\,B}| \cos(\phi)$

$$\cos(\phi) = \frac{{}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,B}}{|{}^{A}\vec{\mathbf{r}}^{\,Q}|\, |{}^{A}\vec{\mathbf{r}}^{\,B}|} = \frac{(7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) \cdot (\text{-}20\,\hat{\mathbf{n}}_z)}{(14.2) * (20)} = \frac{240}{284} \Rightarrow \phi = \text{acos}(\frac{240}{284}) \approx \begin{cases} 0.564 \text{ rad} \\ 32.32^\circ \end{cases}$$

Surface area $\vec{\boldsymbol{\Delta}}$ and unit vector $\hat{\mathbf{u}}$ perpendicular to Δ_{ABQ}.

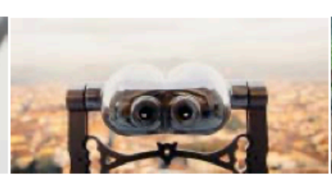
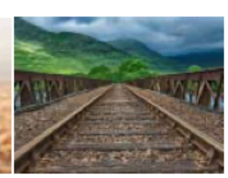
- Visual inspection: Form the position vectors from N_o to Q, N_o to B, and B to A.
  ${}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z \qquad {}^{N_o}\vec{\mathbf{r}}^{\,B} = 8\,\hat{\mathbf{n}}_y \qquad {}^{B}\vec{\mathbf{r}}^{\,A} = 20\,\hat{\mathbf{n}}_z$
- Vector addition: The position from B to Q is: ${}^{B}\vec{\mathbf{r}}^{\,Q} = {}^{B}\vec{\mathbf{r}}^{\,N_o} + {}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$
- The ***vector area*** is: $\vec{\boldsymbol{\Delta}} \underset{(4)}{\triangleq} \tfrac{1}{2}\, {}^{B}\vec{\mathbf{r}}^{\,A} \times {}^{B}\vec{\mathbf{r}}^{\,Q} = \tfrac{1}{2}\,(20\,\hat{\mathbf{n}}_z) \times (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z) = 30\,\hat{\mathbf{n}}_x + 70\,\hat{\mathbf{n}}_y$
- The scalar area is: $|\vec{\boldsymbol{\Delta}}| = \sqrt{\vec{\boldsymbol{\Delta}} \cdot \vec{\boldsymbol{\Delta}}} = \sqrt{30^2 + 70^2} \approx 76.16$
- The unit normal $\hat{\mathbf{u}}$ in the direction of $\vec{\boldsymbol{\Delta}}$ is: $\hat{\mathbf{u}} = \dfrac{\vec{\boldsymbol{\Delta}}}{|\vec{\boldsymbol{\Delta}}|} \approx 0.394\,\hat{\mathbf{n}}_x + 0.919\,\hat{\mathbf{n}}_y$

Distance between point Q and line $\overline{BC}$. $\quad d \underset{(3)}{=} |{}^{B}\vec{\mathbf{r}}^{\,Q} \times \hat{\mathbf{n}}_x| = |8\,\hat{\mathbf{n}}_y - 5\,\hat{\mathbf{n}}_z| = \sqrt{89}$

$+\hat{\mathbf{u}}$ distance from point N_o to plane ABQ. $\quad \delta = \hat{\mathbf{u}} \cdot {}^{N_o}\vec{\mathbf{r}}^{\,B} = (0.394\,\hat{\mathbf{n}}_x + 0.919\,\hat{\mathbf{n}}_y) \cdot 8\,\hat{\mathbf{n}}_y \approx 7.35$

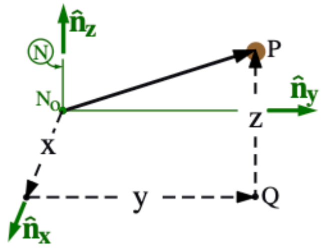
Volume of tetrahedron N_o, A, B, Q. $\quad \text{Volume} \underset{(2.14)}{=} \tfrac{1}{3}\,(\text{-}\vec{\boldsymbol{\Delta}} \cdot {}^{B}\vec{\mathbf{r}}^{\,N_o}) = (\text{-}30\,\hat{\mathbf{n}}_x - 70\,\hat{\mathbf{n}}_y) \cdot (\text{-}8\,\hat{\mathbf{n}}_y) \approx 186.7$



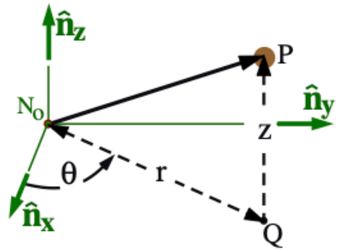



3.5 Measuring position: Cartesian, cylindrical, spherical coordinates

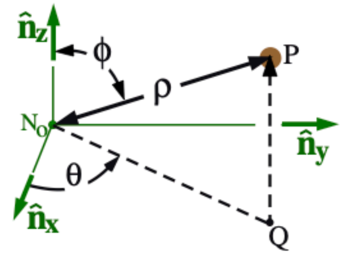
Measuring position: To measure a point P's position usually requires "equipment" e.g., a point N_o, a 3D vector basis $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ from which measurements are made, and 3 scalars such as shown below.



Cartesian coordinates x, y, z



Polar coordinates r, θ, z



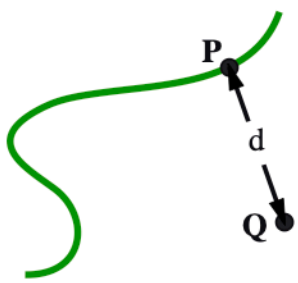
Spherical coordinates ρ, θ, ϕ

Derivatives of position vectors with Cartesian, polar, and spherical coordinate systems are in Homeworks 5.21 to 5.31.

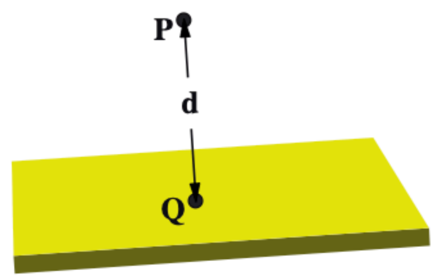
Note: Speed and distance-traveled is in Section 10.8.

3.6 More distance (also see speed and distance-traveled in Section 10.8)

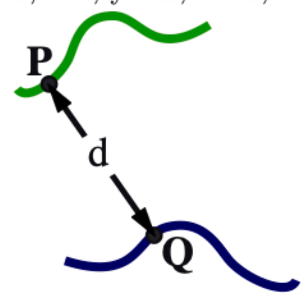
Distance is a non-negative measure of the amount of space separating two points. For example, the figures below show two points P and Q that are separated by a distance d. Distances are measured in the SI system in meters (cm, mm, km, etc.) and measured in the U.S. system in inches, feet, yards, miles, etc.



Distance between a point and a curve



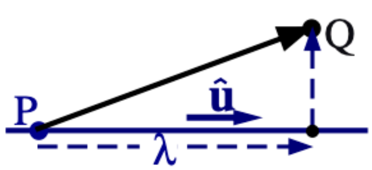
Distance between a point and a plane



Distance between two curves

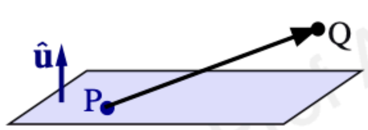
Although distances are non-negative quantities, a ***measure*** may be negative or positive as it is associated with a ***sense***. The ***distance between a point and a curve*** is defined as the distance from the point to the closest point on the curve. The ***distance between a point and a plane*** is the distance from the point to the closest point on the plane. The ***distance between two curves*** is defined to be the distance between the two closest points on the curves. ***Arc-length*** (distance along a curve) is defined as the limit of the sum of distances between sequential points on a curve (as the points get closer together). Section 10.8 discusses the relationship between speed and distance-traveled.

3.7 Optional: Vector equation of a line or plane



Consider a line parallel to a unit vector $\hat{\mathbf{u}}$ and passing through a point P. Eqn (5) gives two ways to ensure an arbitrary point Q lies on the line.

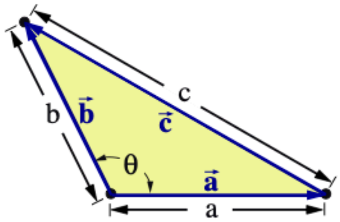
$$\boxed{{}^{P}\vec{\mathbf{r}}^{Q} \times \hat{\mathbf{u}} = \vec{\mathbf{0}} \quad \text{or} \quad {}^{P}\vec{\mathbf{r}}^{Q} = \lambda\,\hat{\mathbf{u}}} \quad \text{where } \lambda = \hat{\mathbf{u}} \cdot {}^{Q}\vec{\mathbf{r}}^{R} \text{ is the } \hat{\mathbf{u}} \text{ measure of } Q\text{'s position from } P. \qquad (5)$$



Consider a plane perpendicular to a unit vector $\hat{\mathbf{u}}$ and passing through a point P. Eqn (6) ensures an arbitrary point Q lies on the plane.

$$\boxed{{}^{P}\vec{\mathbf{r}}^{Q} \cdot \hat{\mathbf{u}} = 0} \qquad (6)$$

3.8 Optional: Proof of law of cosines and law of sines with vectors

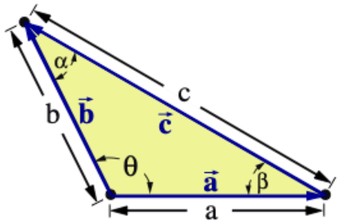


Law of cosines: Proof with definition of vector dot-product [eqn (2.2)].

$$\vec{\mathbf{c}} = -\vec{\mathbf{a}} + \vec{\mathbf{b}}$$

$$c^2 = \vec{\mathbf{c}} \cdot \vec{\mathbf{c}} = (-\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (-\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} - 2\,\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$$

$$\boxed{c^2 = a^2 + b^2 - 2\,a\,b\,\cos(\theta)}$$



Law of sines: Proof with definition of vector cross-product [eqn (2.5)].

$$\left.\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &\triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\sin(\theta)\,\hat{\mathbf{u}} = a\,b\,\sin(\theta)\,\hat{\mathbf{u}} \\ \vec{\mathbf{c}} \times -\vec{\mathbf{a}} &\triangleq |\vec{\mathbf{c}}|\,|-\vec{\mathbf{a}}|\,\sin(\beta)\,\hat{\mathbf{u}} = a\,c\,\sin(\beta)\,\hat{\mathbf{u}} \\ -\vec{\mathbf{b}} \times -\vec{\mathbf{c}} &\triangleq |-\vec{\mathbf{c}}|\,|-\vec{\mathbf{b}}|\,\sin(\alpha)\,\hat{\mathbf{u}} = b\,c\,\sin(\alpha)\,\hat{\mathbf{u}} \end{aligned}\right\}$$

Shown next: Each of these is equal to $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$

$$\vec{\mathbf{c}} \times -\vec{\mathbf{a}} = (-\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times -\vec{\mathbf{a}} = \vec{\mathbf{b}} \times -\vec{\mathbf{a}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

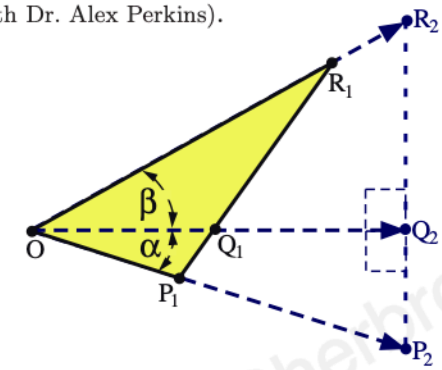
$$-\vec{\mathbf{b}} \times -\vec{\mathbf{c}} = -\vec{\mathbf{b}} \times (\vec{\mathbf{a}} - \vec{\mathbf{b}}) = -\vec{\mathbf{b}} \times \vec{\mathbf{a}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

Since all the previous cross-products equate to $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = a\,b\,\sin(\theta)\,\hat{\mathbf{u}} = a\,c\,\sin(\beta)\,\hat{\mathbf{u}} = b\,c\,\sin(\alpha)\,\hat{\mathbf{u}} \quad \Rightarrow \quad \boxed{\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\theta)}{c}}$$

3.9 Optional: Proof of sine addition formula with vector cross-products

The following figure shows a triangle $\Delta_{OP_1Q_1}$ that has one of its angles divided into angles α and β. Two right-triangles, namely $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$, are constructed as a geometrical starting point for this proof and as a means to provide definitions for $\cos(\alpha)$, and $\cos(\beta)$ (proved with Dr. Alex Perkins).



The areas of triangles $\Delta_{OP_2R_2}$, $\Delta_{OP_2Q_2}$ and $\Delta_{OQ_2R_2}$ are

$$\text{Area}\,\Delta_{O\,P_2\,R_2} = \tfrac{1}{2}\,|\vec{\mathbf{OP_2}} \times \vec{\mathbf{OR_2}}| = \tfrac{1}{2}\,|\vec{\mathbf{OP_2}}|\,|\vec{\mathbf{OR_2}}|\,\sin(\alpha+\beta)$$

$$\text{Area}\,\Delta_{O\,P_2\,Q_2} = \tfrac{1}{2}\,|\vec{\mathbf{OP_2}} \times \vec{\mathbf{OQ_2}}| = \tfrac{1}{2}\,|\vec{\mathbf{OP_2}}|\,|\vec{\mathbf{OQ_2}}|\,\sin(\alpha)$$

$$\text{Area}\,\Delta_{O\,Q_2\,R_2} = \tfrac{1}{2}\,|\vec{\mathbf{OQ_2}} \times \vec{\mathbf{OR_2}}| = \tfrac{1}{2}\,|\vec{\mathbf{OQ_2}}|\,|\vec{\mathbf{OR_2}}|\,\sin(\beta)$$

Equating $\text{Area}\,\Delta_{OP_2R_2} = \text{Area}\,\Delta_{OP_2Q_2} + \text{Area}\,\Delta_{OQ_2R_2}$ and using the definitions of $\cos(\alpha)$ and $\cos(\beta)$ gives

$$|\vec{\mathbf{OP_2}}|\,|\vec{\mathbf{OR_2}}|\,\sin(\alpha+\beta) = |\vec{\mathbf{OP_2}}|\,|\vec{\mathbf{OQ_2}}|\,\sin(\alpha) + |\vec{\mathbf{OQ_2}}|\,|\vec{\mathbf{OR_2}}|\,\sin(\beta)$$

$$\sin(\alpha+\beta) = \frac{|\vec{\mathbf{OQ_2}}|}{|\vec{\mathbf{OR_2}}|}\,\sin(\alpha) + \frac{|\vec{\mathbf{OQ_2}}|}{|\vec{\mathbf{OP_2}}|}\,\sin(\beta)$$

$$\boxed{\sin(\alpha+\beta) = \cos(\beta)\,\sin(\alpha) + \cos(\alpha)\,\sin(\beta)} \tag{7}$$

3.10 Optional: Proof of cosine addition formula via sine addition formula

The proof of the cosine addition formula uses the sine addition formula (proved in Section 3.9) several times.

$$\sin(\underbrace{\alpha+\beta} + 90^\circ) = \sin(\alpha+\beta)\,\cancelto{0}{\cos(90^\circ)} + \cancelto{1}{\sin(90^\circ)}\,\cos(\alpha+\beta) = \cos(\alpha+\beta)$$

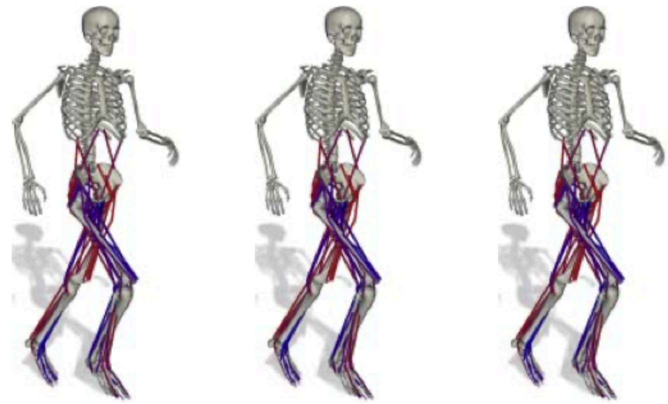
$$\sin(\alpha + \underbrace{\beta + 90^\circ}) = \sin(\alpha)\,\underbrace{\cos(\beta+90^\circ)}_{-\sin(\beta)} + \underbrace{\sin(\beta+90^\circ)}_{\cos(\beta)}\,\cos(\alpha) = -\sin(\alpha)\,\sin(\beta) + \cos(\alpha)\,\cos(\beta)$$

Setting $\alpha = 0$ in the 1^{st} equation gives $\sin(\beta + 90^\circ) = \cos(\beta)$.
Setting $\alpha = 90^\circ$ in the 1^{st} equation gives $\sin(90^\circ + \beta + 90^\circ) = \cos(\beta + 90^\circ)$. $\alpha = 180^\circ$ in eqn (7) gives $\sin(180^\circ + \beta) = -\sin(\beta)$.
The equations in the previous line combine to $\sin(90^\circ + \beta + 90^\circ) = \cos(\beta + 90^\circ) = -\sin(\beta)$

$$\boxed{\cos(\alpha+\beta) = \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta)}$$

Chapter 4

Vector basis

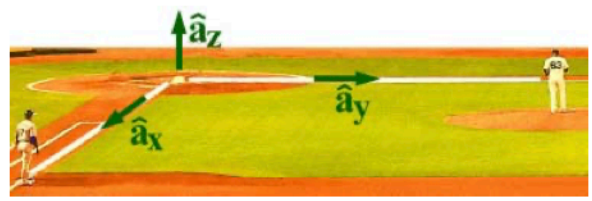


Courtesy Dr. Sam Hamner

Why use a vector basis? (see examples in Hw 1, 2, 3)

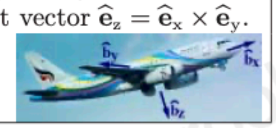
Unit vectors are sign-posts (up, down, left, right, North, East, etc.). A ***vector basis*** with 3 orthogonal unit vectors provide a way to "give directions" in 3D (three-dimensional) space. Conventions for directions depend on the analyst and field of study, e.g., biomechanics, aeronautics, vehicle dynamics, statics, etc.

The vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ shown right form a 3D vector basis. The basis is a ***right-handed orthogonal unitary basis***.



- ***Right-handed*** (***dextral***) because $\widehat{\mathbf{a}}_x \times \widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{a}}_z > 0$.
- ***Orthogonal*** because of the 90° angles between $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$.
- ***Unitary basis*** because $|\widehat{\mathbf{a}}_x| = |\widehat{\mathbf{a}}_y| = |\widehat{\mathbf{a}}_z| = 1$.

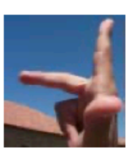
Examples of 3D unitary vector bases used for vehicle dynamics, aerospace vehicles, ...

NED	Used for Earth's **surface** with unit vectors directed locally $\widehat{\mathbf{N}}$orth, $\widehat{\mathbf{E}}$ast, and $\widehat{\mathbf{D}}$own.
ENU	Used for Earth's **surface** with unit vectors directed locally $\widehat{\mathbf{E}}$ast, $\widehat{\mathbf{N}}$orth, and $\widehat{\mathbf{U}}$p.
ECEF	**E**arth-**C**entered/**E**arth-**F**ixed basis orients spacecraft and other celestial objects relative to Earth, with a unit vector $\widehat{\mathbf{e}}_x$ pointing from Earth's center to 0 latitude (equator) 0 longitude (Greenwich prime meridian), a second unit vector $\widehat{\mathbf{e}}_y$ pointing to geometric North, and a third unit vector $\widehat{\mathbf{e}}_z = \widehat{\mathbf{e}}_x \times \widehat{\mathbf{e}}_y$.
$\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$	Unit vectors to orient a generic body B. For example, one way to orient an aircraft is with: aircraft forward $\widehat{\mathbf{b}}_x$, pilot right $\widehat{\mathbf{b}}_y$, and pilot down $\widehat{\mathbf{b}}_z = \widehat{\mathbf{b}}_x \times \widehat{\mathbf{b}}_y$. 

4.1 What is a vector basis?

A ***vector basis*** is a set of linearly independent vectors that span a space (e.g., the 3D space in which we live). Each linearly independent vector is called a ***basis vector*** for the space. Shown right are three pictures of the same ***3D right-handed orthogonal unitary basis***.

Notice: $\widehat{\mathbf{a}}_x \times \widehat{\mathbf{a}}_y = \widehat{\mathbf{a}}_z$, $\widehat{\mathbf{a}}_y \times \widehat{\mathbf{a}}_z = \widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_z \times \widehat{\mathbf{a}}_x = \widehat{\mathbf{a}}_y$ (when $\widehat{\mathbf{a}}_z$ is absent, it is implied by the ***right-hand rule***).



To physically demonstrate an orthogonal vector basis, hold your right hand with the thumb, forefinger, and middle finger pointing in orthogonal directions. Chapter 5 deals with ***rotation matrix*** and is summarized with two right hands (each with a vector basis) and the question "how do I relate two vector bases".

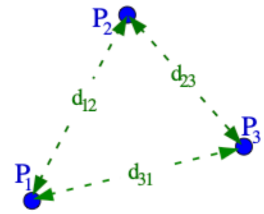
Expressing a 3D vector basis in terms of another 3D vector basis (also see Chapter 5)

Basis vectors $\vec{\mathbf{b}}_1$, $\vec{\mathbf{b}}_2$, $\vec{\mathbf{b}}_3$ can be expressed in terms of basis vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ with the scalar functions $\mathbb{R}_{ij}$ $(i, j = 1, 2, 3)$.

$$\begin{aligned}\vec{\mathbf{b}}_1 &= \mathbb{R}_{11}\,\vec{\mathbf{a}}_1 + \mathbb{R}_{12}\,\vec{\mathbf{a}}_2 + \mathbb{R}_{13}\,\vec{\mathbf{a}}_3\\ \vec{\mathbf{b}}_2 &= \mathbb{R}_{21}\,\vec{\mathbf{a}}_1 + \mathbb{R}_{22}\,\vec{\mathbf{a}}_2 + \mathbb{R}_{23}\,\vec{\mathbf{a}}_3\\ \vec{\mathbf{b}}_3 &= \mathbb{R}_{31}\,\vec{\mathbf{a}}_1 + \mathbb{R}_{32}\,\vec{\mathbf{a}}_2 + \mathbb{R}_{33}\,\vec{\mathbf{a}}_3\end{aligned} \quad \text{or} \quad \begin{bmatrix}\vec{\mathbf{b}}_1\\ \vec{\mathbf{b}}_2\\ \vec{\mathbf{b}}_3\end{bmatrix} = \begin{bmatrix}\mathbb{R}_{11} & \mathbb{R}_{12} & \mathbb{R}_{13}\\ \mathbb{R}_{21} & \mathbb{R}_{22} & \mathbb{R}_{23}\\ \mathbb{R}_{31} & \mathbb{R}_{32} & \mathbb{R}_{33}\end{bmatrix}\begin{bmatrix}\vec{\mathbf{a}}_1\\ \vec{\mathbf{a}}_2\\ \vec{\mathbf{a}}_3\end{bmatrix}$$

When both $\vec{\mathbf{a}}_i$ and $\vec{\mathbf{b}}_i$ $(i = 1, 2, 3)$ are right-handed orthogonal unitary bases, the matrix relating $\vec{\mathbf{b}}_i$ to $\vec{\mathbf{a}}_i$ is called the ${}^{b}R^{a}$ ***rotation matrix*** and has many special properties as described in Chapter 5.

4.2 Rigid and non-rigid bases



When the magnitude of each basis vector in a vector basis is **constant** and the angles between bases vectors are **constant**, the basis is a "***rigid vector basis***".

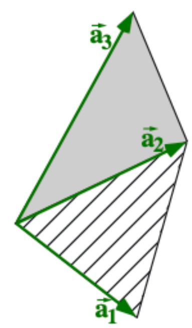
When the magnitude of a basis vector in a vector basis is **variable** or an angle between two basis vectors are **variable**, the basis is a "***non-rigid vector basis***".

For example, the previous figure shows distinct non-collinear points P_1, P_2, P_3 and the non-zero distances between them d_{12}, d_{23}, d_{31}. One way to construct a basis is from the vector $\vec{\mathbf{a}}_1$ directed from P_1 to P_2, the vector $\vec{\mathbf{a}}_2$ directed from P_1 to P_3, and $\vec{\mathbf{a}}_3 = \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2$. This is a ***rigid vector basis*** if all the distances are constant whereas it is a ***non-rigid vector basis*** if d_{12} is variable.

Note: Even though a ***rigid vector basis*** can sometimes be associated with a unique ***reference frame***, a reference frame contains an infinite number of vector bases. For example, a rigid basis consisting of $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ may be fixed in a reference frame A. However, one is free to fix other rigid bases (e.g., $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$) in A.

Note: A ***reference frame*** is a ***rigid object*** that can be constructed with as few as three non-collinear points whose distance from each other are constant. Reference frames are discussed in Chapter 8 and differ from a rigid basis in that at least one **point** must be fixed in a reference frame whereas a rigid basis is not associated with a point. Differences between reference frames and rigid basis are most relevant when dealing with **translational** kinematics (e.g., velocity and acceleration).

4.3 Non-orthogonal 3D vector basis



Shown right, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ is a ***right-handed, non-orthogonal, non-unitary basis***.

- ***Right-handed*** (***dextral***) because $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 > 0$.
- ***Non-orthogonal*** since the angle between each of $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ is not 90°.
  If $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ were **orthogonal** $\vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_2 = 0$, $\vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_3 = 0$, $\vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 = 0$.
- ***Non-unitary*** because $|\vec{\mathbf{a}}_i| \neq 1$ ($i = 1$ or $i = 2$ or $i = 3$).

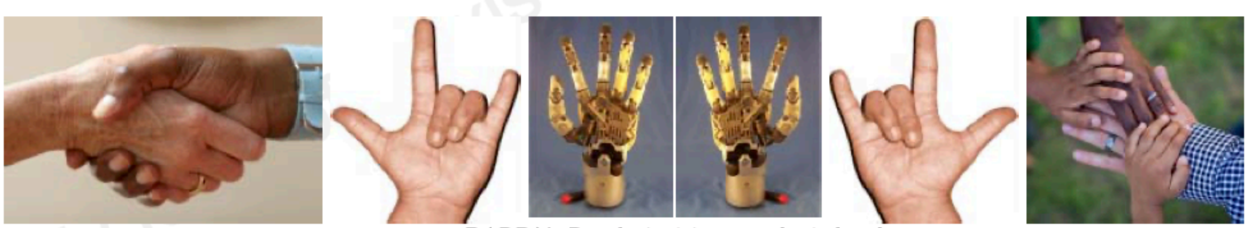
Non-orthogonal bases facilitate certain calculations. For example, they play an important role in determining the volume, centroid, and inertia properties of a tetrahedron, a shape used by CAD/CAE programs for constructing nearly any geometrical object.

Non-orthogonal bases are useful in motion studies (e.g., ***gait studies***) involving irregularly-shaped objects (e.g., human bones) that require ***markers*** (devices which track a point's location) on easily-identifiable, physically-meaningful locations (e.g., ***anatomic landmarks***). It is easier (and physically meaningful) to construct a non-orthogonal basis with vectors associated with markers (e.g., pointing from one marker to another marker).

4.4 Creating various 3D bases from two non-parallel vectors

There are various ways to construct a 3D vector basis from two non-parallel vectors $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$. For example, one way is to form $\vec{\mathbf{a}}_3 \triangleq \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2$, and then use

Non-orthogonal:	$\vec{\mathbf{a}}_1$	$\vec{\mathbf{a}}_2$	$\vec{\mathbf{a}}_3 \triangleq \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2$
Orthogonal:	$\vec{\mathbf{a}}_1$	$\vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1$	$\vec{\mathbf{a}}_3$
Orthogonal, unitary:	$\widehat{\mathbf{a}}_x = \frac{\vec{\mathbf{a}}_1}{\lvert\vec{\mathbf{a}}_1\rvert}$	$\widehat{\mathbf{a}}_y = \frac{\vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1}{\lvert\vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1\rvert}$	$\widehat{\mathbf{a}}_z = \frac{\vec{\mathbf{a}}_3}{\lvert\vec{\mathbf{a}}_3\rvert}$

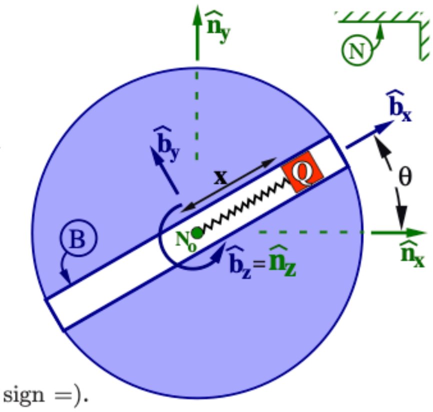


DARPA's Revolutionizing prosthetic hand.
Courtesy of HDT Engineering Services and Kinea Design.

4.5 Concept: What is the vector vs. how is it expressed

Shown right is a particle Q sliding along a straight track/body B. Track B spins in reference frame N. A spring (with linear spring constant k) connects Q to point N_o (N_o is **stationary** in both B and N). Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in B and N, respectively with:

- $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ parallel to B's axis of rotation in N.
- $\widehat{\mathbf{b}}_x$ directed along the track from N_o to Q and $\widehat{\mathbf{n}}_x$ horizontally-right.



Using geometry, $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ relate to $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ as

$$\begin{aligned} \widehat{\mathbf{b}}_x &= \cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y \\ \widehat{\mathbf{b}}_y &= \text{-}\sin(\theta)\,\widehat{\mathbf{n}}_x + \cos(\theta)\,\widehat{\mathbf{n}}_y \\ \widehat{\mathbf{b}}_z &= \widehat{\mathbf{n}}_z \end{aligned} \quad \Rightarrow$$

	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$\text{-}\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

The examples below clarify two distinct concepts:

- **What is the vector** (the name to the **left** of the equal sign =).
- **How is the vector expressed** (the expression on the **right** of the equal sign =).

$\vec{\mathbf{F}}$ (the spring force on Q) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.
$\vec{\mathbf{F}} = \text{-}k\,x\,\widehat{\mathbf{b}}_x$

Expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
$\vec{\mathbf{F}} = \text{-}k\,x\,[\cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y]$

$$\vec{\mathbf{F}} = \begin{bmatrix} \text{-}kx \\ 0 \\ 0 \end{bmatrix}_{\widehat{\mathrm{b}}_{xyz}} = \begin{bmatrix} \text{-}kx\cos(\theta) \\ \text{-}kx\sin(\theta) \\ 0 \end{bmatrix}_{\widehat{\mathrm{n}}_{xyz}}$$

${}^{N_o}\vec{\mathbf{r}}^{\,Q}$ (Q's position from N_o) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.
${}^{N_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x$

Expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
${}^{N_o}\vec{\mathbf{r}}^{\,Q} = x\,[\cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y]$

$${}^{N_o}\vec{\mathbf{r}}^{\,Q} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}_{\widehat{\mathrm{b}}_{xyz}} = \begin{bmatrix} x\cos(\theta) \\ x\sin(\theta) \\ 0 \end{bmatrix}_{\widehat{\mathrm{n}}_{xyz}}$$

${}^{B}\vec{\mathbf{v}}^{\,Q}$ (Q's velocity in **B**) can be ***expressed*** in various bases.

Expressed in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.
${}^{B}\vec{\mathbf{v}}^{\,Q} = \dot{x}\,\widehat{\mathbf{b}}_x$

Expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
${}^{B}\vec{\mathbf{v}}^{\,Q} = \dot{x}\,[\cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y]$

$${}^{B}\vec{\mathbf{v}}^{\,Q} = \begin{bmatrix} \dot{x} \\ 0 \\ 0 \end{bmatrix}_{\widehat{\mathrm{b}}_{xyz}} = \begin{bmatrix} \dot{x}\cos(\theta) \\ \dot{x}\sin(\theta) \\ 0 \end{bmatrix}_{\widehat{\mathrm{n}}_{xyz}}$$

${}^{N}\vec{\mathbf{v}}^{\,Q}$ (Q's velocity in **N**) can be ***expressed*** in various bases. Note: ${}^{N}\vec{\mathbf{v}}^{\,Q} \neq {}^{B}\vec{\mathbf{v}}^{\,Q}$. These are **different** vectors.

Expressed in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.
${}^{N}\vec{\mathbf{v}}^{\,Q} = \dot{x}\,\widehat{\mathbf{b}}_x + x\,\dot{\theta}\,\widehat{\mathbf{b}}_y$

Expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
${}^{N}\vec{\mathbf{v}}^{\,Q} = [\dot{x}\cos(\theta) - x\,\dot{\theta}\sin(\theta)]\,\widehat{\mathbf{n}}_x + [\dot{x}\sin(\theta) + x\,\dot{\theta}\cos(\theta)]\,\widehat{\mathbf{n}}_y$

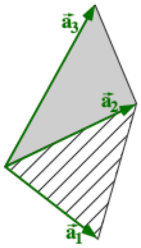
Remember: A vector is not changed by ***expressing*** it in a different basis.

Advanced/ahead in Chapter 7: Vector differentiation depends on rigid basis (reference frame).

The vector $\vec{\mathbf{r}} = x\,\widehat{\mathbf{b}}_x$ is not changed by how it is expressed. However, $\vec{\mathbf{r}}$'s **derivative** in rigid basis **B** (denoted $\frac{{}^{B}d\vec{\mathbf{r}}}{dt}$) is **different** than $\vec{\mathbf{r}}$'s **derivative** in rigid basis **N** (denoted $\frac{{}^{N}d\vec{\mathbf{r}}}{dt}$). In other words, $\frac{{}^{N}d\vec{\mathbf{r}}}{dt} \neq \frac{{}^{B}d\vec{\mathbf{r}}}{dt}$.

$$\begin{aligned} \frac{{}^{B}d\vec{\mathbf{r}}}{dt} &= \dot{x}\,\widehat{\mathbf{b}}_x &&= \dot{x}\,[\cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y] \\ \frac{{}^{N}d\vec{\mathbf{r}}}{dt} &= \dot{x}\,\widehat{\mathbf{b}}_x + x\,\dot{\theta}\,\widehat{\mathbf{b}}_y &&= [\dot{x}\cos(\theta) - x\,\dot{\theta}\sin(\theta)]\,\widehat{\mathbf{n}}_x + [\dot{x}\sin(\theta) + x\,\dot{\theta}\cos(\theta)]\,\widehat{\mathbf{n}}_y \end{aligned}$$

4.6 Expressing a vector in terms of the basis $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$



Given an arbitrary vector $\vec{\mathbf{v}}$ and **3** non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, one can ***express*** $\vec{\mathbf{v}}$ in terms of vector basis $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ and scalar quantities v_1, v_2, v_3 (e.g., numbers or functions of time t) as[†]

$$\boxed{\vec{\mathbf{v}} = v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3} \tag{1}$$

$$v_1 = \frac{\vec{\mathbf{v}} \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3)}{\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3)} \qquad v_2 = \frac{\vec{\mathbf{v}} \cdot (\vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1)}{\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3)} \qquad v_3 = \frac{\vec{\mathbf{v}} \cdot (\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2)}{\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3)} \tag{2}$$

When $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are **orthogonal unit** vectors, equations (1) and (2) simplify to

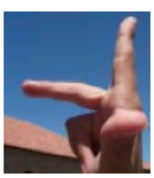
$$\vec{\mathbf{v}} \underset{(1)}{=} v_x\,\hat{\mathbf{a}}_x + v_y\,\hat{\mathbf{a}}_y + v_z\,\hat{\mathbf{a}}_z \qquad v_x \underset{(2)}{=} \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_x \qquad v_y \underset{(2)}{=} \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_y \qquad v_z \underset{(2)}{=} \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_z \tag{3}$$

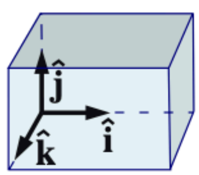
[†] To prove v_1 in eqn (2), dot multiply both sides of eqn (1) with $\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3$ to get $\vec{\mathbf{v}} \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3) \underset{(1)}{=} v_1\,\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3)$. Isolate v_1 to prove the first expression in eqn (2). Proceed similarly to find v_2 and v_3 and use the Section 2.11 scalar triple product property $\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3) = \vec{\mathbf{a}}_2 \cdot (\vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1) = \vec{\mathbf{a}}_3 \cdot (\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2)$. Note: Since $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are non-parallel, non-coplanar $\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3) \neq 0$.

Projections, ***measures***, ***coefficients***, ***components***
The ***projection*** or ***measure*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$ is defined in Section 2.9.2 as $\vec{\mathbf{v}} \cdot \frac{\vec{\mathbf{b}}}{\lvert\vec{\mathbf{b}}\rvert}$
Coefficients require context in a mathematical expression. Consider the coplanar vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$. With $\vec{\mathbf{v}} = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y$, the coefficient of $\hat{\mathbf{j}}$ is 3 whereas the coefficient of $\hat{\mathbf{a}}_y$ is y.
Components are **vectors**. The $\hat{\mathbf{a}}_x$ component in $\vec{\mathbf{v}}$ is the ***coefficient*** of $\hat{\mathbf{a}}_x$ multiplied by $\hat{\mathbf{a}}_x$, e.g., $x\,\hat{\mathbf{a}}_x$.

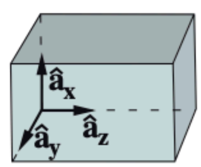
4.7 Right-handed (standard) and left-handed (unconventional) bases

A set of 3 vectors with intrinsic order, e.g., $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, is called ***right-handed*** when $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 > 0$, whereas the set is ***left-handed*** if $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3 < 0$. It is **conventional** to use a ***right-handed basis***.

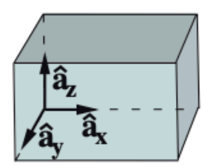




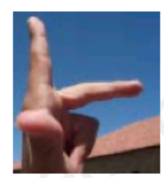
Right-handed basis



Right-handed basis



Left-handed basis

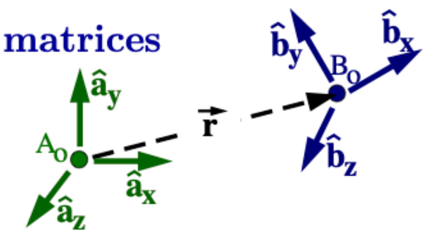


Optional: The language, history, and culture of "left" and "right"

Language	Word	Translation	Meaning	More info
English	right	right	correct, "you are right"	Engineers like being "right"
English	left	left	"left out"	
French	right	droit	adroit means to the right or skillful	
French	left	gauche	socially clumsy	http://www.gauche.com
Latin	right	dexter	nimble, dexterous	
Latin	left	sinistre	dark and mysterious	
Greek	right	orthos	root of the word orthogonal	Also dexion
Greek	left	skalos	awkward, ill-omened	

Famous left-handers include Alexander the Great, Julius Caesar, Leonardo, Michelangelo, Raphael, Newton, Curie, Henry Ford, Napoleon, and a disproportion of Nobel-prize winners and recent U.S. presidents, including Gerald Ford, Ronald Regan (ambidextrous), Bill Clinton, and Barack Obama. One ultrasound study showed 90% of in-utero babies sucking their right thumb. In a study of 100,000 students taking the SAT (Scholastic Aptitude Test), 20% of the top-scoring group was left-handed, twice the 10% rate of left-handed students in the general population. In 2007, gene LRRTM1 was associated with both left-handedness and an increased chance of schizophrenia.

4.8 Optional: Rigid frames and transformation matrices



A ***rigid frame*** is the combination of a ***rigid vector basis*** and an ***origin*** point. Shown right are orthonormal ***rigid frames*** A and B.

- A has right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and origin A_o.
- B has right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ and origin B_o.

Rigid frames are used in robotics, computer graphics, multibody dynamics, etc., and are a convenient tool to measure a rigid body's ***configuration (pose)***, i.e., its orientation and position relative to another rigid body or frame.

The ***transformation matrix*** ${}^AX^B$ relates the orientation and position of rigid frames A and B. It has a 3×3 ${}^aR^b$ ***rotation matrix*** (described in Chapter 5) and a 3×1 ***position matrix*** containing the $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ measures of B_o's position from A_o.

$$ {}^AX^B \triangleq \left[\begin{array}{ccc|c} & {}^aR^b & & \left[{}^{A_o}\vec{\mathbf{r}}^{B_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad (4)$$

To obey matrix multiplication rules ${}^AX^B$ can be regarded as a 4×4 matrix whose last row is $[0 \;\; 0 \;\; 0 \;\; 1]$. For efficient storage and computation, modern methods use a 3×4 ***transformation matrix/table***.

3×4 transformation matrix

$$ {}^AX^B = \underbrace{\begin{bmatrix} \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_x & \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_y & \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_z \\ \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_x & \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_y & \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_z \\ \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_x & \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_y & \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_z \end{bmatrix}}_{{}^aR^b} \Bigg| \underbrace{\begin{matrix} \hat{\mathbf{a}}_x\cdot{}^{A_o}\vec{\mathbf{r}}^{B_o} \\ \hat{\mathbf{a}}_y\cdot{}^{A_o}\vec{\mathbf{r}}^{B_o} \\ \hat{\mathbf{a}}_z\cdot{}^{A_o}\vec{\mathbf{r}}^{B_o} \end{matrix}}_{\left[{}^{A_o}\vec{\mathbf{r}}^{B_o}\right]_a} $$

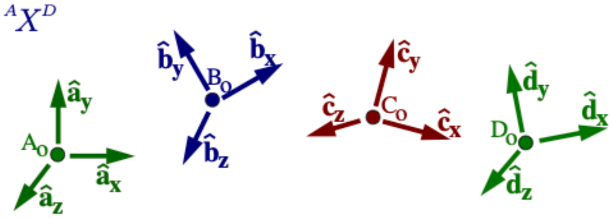
3×4 transformation table

${}^AX^B$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$	${}^{A_o}\vec{\mathbf{r}}^{B_o}$
$\hat{\mathbf{a}}_x$	0.696	-0.717	0.050	4.2
$\hat{\mathbf{a}}_y$	0.710	0.697	0.010	3.3
$\hat{\mathbf{a}}_z$	-0.106	-0.034	0.994	-1.2

Numerical example of a ***transformation table***.

Concatenating to form transformation matrix ${}^AX^D$

${}^AX^D$ can be computed by concatenating transformation matrices associated with rigid frames A, B, C, D.



$$ {}^AX^D = {}^AX^B * {}^BX^C * {}^CX^D $$

When ${}^AX^D$ is a 4×4 matrix with 4^{th} row $[0\;0\;0\;1]$, the previous equation obeys the rules of matrix multiplication.

Inverse of transformation matrix ${}^AX^B$

The inverse of ${}^AX^B$ is defined as ${}^BX^A$.

Note: When A and B are orthonormal, the inverse of ${}^AX^B$ can be efficiently calculated via the inverse-property of orthonormal rotation matrices ${}^bR^a = \left({}^aR^b\right)^{-1} = \left({}^aR^b\right)^T$.

$$ \left({}^AX^B\right)^{-1} \triangleq {}^BX^A \underset{(4)}{=} \left[\begin{array}{ccc|c} & {}^bR^a & & -{}^bR^a * \left[{}^{A_o}\vec{\mathbf{r}}^{B_o}\right]_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

As a 3×4 matrix, ${}^BX^A$ does not have the 4^{th} row $[0\;0\;0\;1]$.

Position vector to an arbitrary point Q using ${}^AX^B$

As shown right, Q's position from A_o expressed in A can be calculated from ${}^AX^B$ and Q's position from B_o expressed in B.

$$ \left[{}^{A_o}\vec{\mathbf{r}}^{Q}\right]_a = {}^AX^B * \left[{}^{B_o}\vec{\mathbf{r}}^{Q}\right]_b $$

Note: To obey rules of matrix multiplication, if ${}^AX^B$ has 4^{th} row $[0\;0\;0\;1]$, append a 4^{th} element with value 1 to each position matrix.

Vocabulary: All the previous transformation matrices are ***proper rigid transforms (isometries)*** and can rotate and translate a rigid body/frame without changing its shape or size (**rigid**) and without mirroring/reflecting the body (**proper**). Rigid transforms are a special type of ***similarity transform*** which is any combination of translation, rotation, reflection, and scaling (e.g., similar triangles). Similarity transforms are a subset of ***affine transform*** which also include reflection and shear. The set of all rigid transforms is the ***Euclidean group***, denoted E(n) for n-dimensional Euclidean space. The set of all proper rigid transforms in 3D-space is ***SE***(3) (**S**pecial **E**uclidean 3D space – the 3D world in which we live). The set of all proper rigid rotations in 3D-space is ***SO***(3) (**S**pecial **O**rthogonal 3D rotations).

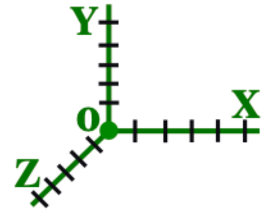
Note: "Transform" can be regarded as a noun or verb.

- **Transform as a noun**: ${}^AX^B$ describes B's orientation and position in A.
- **Transform as a verb**: ${}^AX^B$ rotates and translates B relative to A.

4.9 Optional: Coordinate system vs. vector bases

A ***coordinate system*** is a **set** of scalar quantities, typically angles or distances, used in specifying the location of points, curves, surfaces, and solids. A ***coordinate*** is a **single** scalar in the set.[1] A ***generalized coordinate*** is a scalar quantity that is useful in locating points, curves, surfaces, and solids but is not necessarily associated with a coordinate system such as a Cartesian, cylindrical, or spherical[2] coordinate system.[3] In other words a generalized coordinate is a more general type of coordinate. Generalized coordinates play an increasingly important role in geometry and motion.

Coordinate system	Method for locating points
Cartesian coordinate system	3 distances measures, e.g., (x, y, z)
Cylindrical coordinate system	2 distances and 1 angle, e.g., (r, θ, z)
Spherical coordinate system	1 distance and 2 angles, e.g., (ρ, θ, ϕ)



Cartesian coordinate system

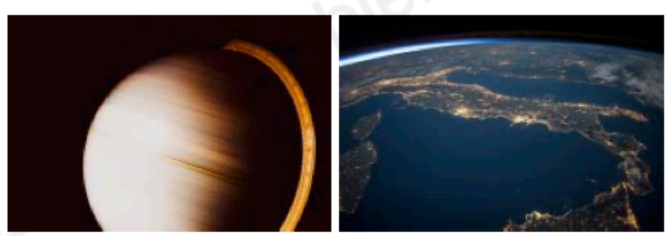
The most famous coordinate system is a rectangular ***Cartesian coordinate system*** which consists of three mutually-perpendicular lines, called ***coordinate axes***, along which measurements are done and which all intersect at one point called the ***origin***. The differences between a Cartesian coordinate system and a vector basis are highlighted below.[a]

[a] **It is usually more efficient to use generalized coordinates and vector bases than coordinate systems.**

Vector basis

- A Cartesian coordinate system has an **origin** and a **set of coordinates**. A basis does not.
- A Cartesian coordinate system has **coordinate axes** along which measurements are done. A basis does not.
- A Cartesian coordinate system does not intrinsically have a basis - although one can easily be constructed by introducing unit vectors that are oriented parallel to the coordinate axes and whose sense is determined by the positive direction along the coordinate axes.

Note: A ***vector basis*** is defined as a linearly independent set of vectors that "***span a space***".
A set of vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ is said to "***span 3D space***" (e.g., Earth's 3D space) if and only if any arbitrary vector $\vec{\mathbf{v}}$ in 3D space can be written as a "***linear combination***" of $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$. A ***linear combination*** of vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ is defined as $\sum_{i=1}^{n} v_i\, \vec{\mathbf{a}}_i$ where v_i are scalar quantities. As shown in Section 4.6, if $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ form a 3D vector basis, any arbitrary 3D vector can be written $\vec{\mathbf{v}} = v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3$. One way to form a basis is by "guess and check". As shown in Section 4.6, one may guess $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ form a 3D basis. The solution for v_1, v_2, v_3 checks if and only if $\vec{\mathbf{a}}_1 \cdot (\vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3) \neq 0$, which means $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are non-coplanar vectors.

[1] A coordinate may be a variable, constant, or specified function of time.

[2] A ***spherical coordinate system*** is useful for describing the location of a point on a sphere. When studying particle motion on Earth's surface (e.g., a particle in a river), a spherical coordinate system can be advantageous because ρ is constant and only two variables (not three) are needed in the analysis. Using a Cartesian coordinate system for particle motion on a sphere's surface requires the "constraint" relationship $x^2 + y^2 + z^2 = $ constant. Similarly, a polar coordinate system requires a constraint relationship $r^2 + z^2 = $ constant. ***However***, Homework 5.29 shows that spherical coordinates have an inherent singularity at the "North" and "South" pole.

[3] Other related coordinates include curvilinear, Plucker, canonical, intrinsic, parallel, elliptic, ellipsoidal, prolate spheroidal, oblate spheroidal, conical, parabolic, paraboloidal, toroidal, bispherical, biangular, etc.