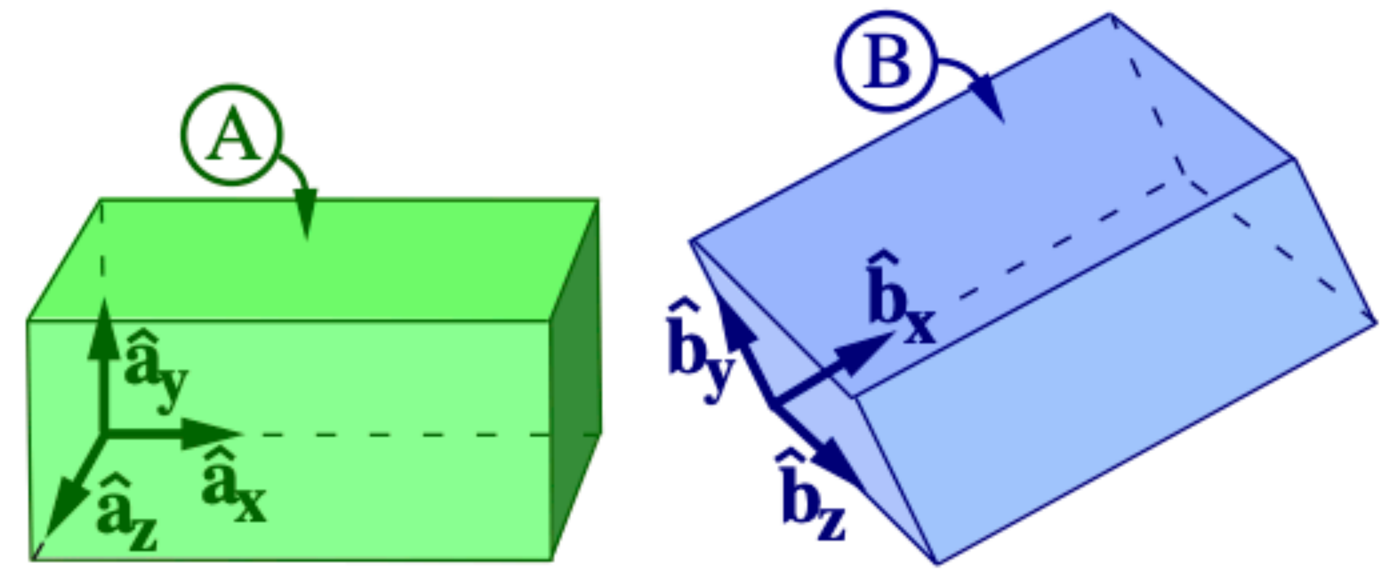
Chapter 5

Rotation matrices I

Examples in Hw 4. Advanced rotation matrices and ODEs in Chapters 6 and 9.



The figure shows two sets of ***right-handed orthogonal unit vectors***, namely a set a with $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and a set b with $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

One way to describe the ***orientation*** between a and b is with a 3×3 ***rotation matrix*** ${}^aR^b$ whose elements are defined in eqn (1) as ${}^aR^b_{ij} \triangleq \widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{b}}_j$ $(i, j = x, y, z)$. Eqn (1) also shows ${}^aR^b_{ij}$ is equal to the cosine of the angle between $\widehat{\mathbf{a}}_i$ and $\widehat{\mathbf{b}}_j$.

$$ {}^aR^b \triangleq \begin{bmatrix} \widehat{\mathbf{a}}_x \\ \widehat{\mathbf{a}}_y \\ \widehat{\mathbf{a}}_z \end{bmatrix} \cdot \begin{bmatrix} \widehat{\mathbf{b}}_x & \widehat{\mathbf{b}}_y & \widehat{\mathbf{b}}_z \end{bmatrix} \qquad {}^aR^b_{ij} \triangleq \widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{b}}_j = \cos\angle(\widehat{\mathbf{a}}_i,\ \widehat{\mathbf{b}}_j) \tag{1} $$

Proof: $\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{b}}_j \underset{(2.2)}{\triangleq} \underbrace{|\widehat{\mathbf{a}}_i|}_{1} \underbrace{|\widehat{\mathbf{b}}_j|}_{1} \cos\angle(\widehat{\mathbf{a}}_i,\ \widehat{\mathbf{b}}_j)$.

A rotation matrix is also called a ***direction cosine matrix*** whose elements are ***direction cosines***. **Advanced:** Chapter 6 shows how orientation can be represented with Euler angles, quaternions (Euler parameters), Rodrigues parameters, etc.

A ***rotation matrix*** R is an orthogonal matrix, which means the transpose of R is equal to the inverse of R, i.e., $R^T = R^{-1}$.
Section 5.4.2 gives an example of calculating the inverse of a rotation matrix.

$$ {}^bR^a = \left({}^aR^b\right)^{-1} = \left({}^aR^b\right)^T \tag{2} $$

Eqn (2) is proved in Section 5.8.1.

Since a rotation matrix is orthogonal (its inverse is its transpose), it is convenient to store orientation information in a ***rotation table*** that can be read horizontally or vertically.[a]

${}^aR^b$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_y$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_x$	$\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_y$	$\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{b}}_x$	$\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{b}}_y$	$\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_z$	$\widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{b}}_x$	$\widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{b}}_y$	$\widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{b}}_z$

(3)

[a] If either basis is non-orthogonal, use a relationship matrix (**not** a rotation matrix) because the inverse of a non-orthogonal matrix is **not** its transpose.

The rotation matrix ${}^aR^d$ can be formed by successive matrix multiplication of the ${}^aR^b$, ${}^bR^c$, ${}^cR^d$ rotation matrices.
Section 5.6 gives an example of successive matrix multiplication.

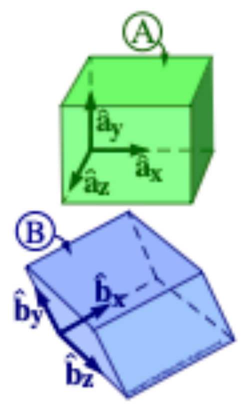
$$ {}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d \tag{4} $$

Squash rule for multiplying rotation matrices

5.1 Uses for the rotation matrix ${}^aR^b$ (for geometry, statics, motion analysis, stress ...)

Dot-product	between unit vectors $\widehat{\mathbf{a}}_i$ and $\widehat{\mathbf{b}}_j$ $(i, j = x, y, z)$, e.g., $\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_y$.
Angle	between unit vectors $\widehat{\mathbf{a}}_i$ and $\widehat{\mathbf{b}}_j$ $(i, j = x, y, z)$, e.g., $\angle(\widehat{\mathbf{a}}_x,\ \widehat{\mathbf{b}}_y)$.
Dot-product	of $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ (arbitrary vectors), each which have $\widehat{\mathbf{a}}_i$ and/or $\widehat{\mathbf{b}}_j$ components (e.g., Section 5.4).
Cross-product	of $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$, each which may be expressed in terms of $\widehat{\mathbf{a}}_i$ and/or $\widehat{\mathbf{b}}_j$.
Expressing	a vector $\vec{\mathbf{v}}$ written in terms of $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ instead in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ (or vice-versa).
Multiplication	to form other rotation matrices, e.g., ${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$ [e.g., Section 5.6, eqn (7)].
Angular velocity	of rigid basis $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ relative to rigid basis $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$.
$[\vec{\mathbf{v}}]_a = {}^aR^b\,[\vec{\mathbf{v}}]_b$ ***Squash rule***	to relate the column matrix representation of vector $\vec{\mathbf{v}}$ expressed in terms of $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ to its column matrix representation in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

5.2 Rotation matrices and "squash rules"

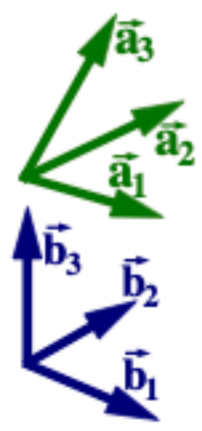


A rotation matrix R relates **column** or **row** matrices of sets of **orthogonal unit** vectors.

$$\begin{bmatrix}\widehat{\mathbf{a}}_x\\ \widehat{\mathbf{a}}_y\\ \widehat{\mathbf{a}}_z\end{bmatrix} = {}^aR^b \begin{bmatrix}\widehat{\mathbf{b}}_x\\ \widehat{\mathbf{b}}_y\\ \widehat{\mathbf{b}}_z\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\widehat{\mathbf{b}}_x\\ \widehat{\mathbf{b}}_y\\ \widehat{\mathbf{b}}_z\end{bmatrix} = {}^bR^a \begin{bmatrix}\widehat{\mathbf{a}}_x\\ \widehat{\mathbf{a}}_y\\ \widehat{\mathbf{a}}_z\end{bmatrix}$$

Transposing and using orthogonality $({}^aR^b)^T \underset{(2)}{=} {}^bR^a$

$$\begin{aligned}[\widehat{\mathbf{a}}_x \;\; \widehat{\mathbf{a}}_y \;\; \widehat{\mathbf{a}}_z] &= [\widehat{\mathbf{b}}_x \;\; \widehat{\mathbf{b}}_y \;\; \widehat{\mathbf{b}}_z]\,{}^bR^a \\ [\widehat{\mathbf{b}}_x \;\; \widehat{\mathbf{b}}_y \;\; \widehat{\mathbf{b}}_z] &= [\widehat{\mathbf{a}}_x \;\; \widehat{\mathbf{a}}_y \;\; \widehat{\mathbf{a}}_z]\,{}^aR^b\end{aligned} \qquad (5)$$



A relationship matrix $\mathbb{R}$ relates **column** or **row** matrices of **non-orthogonal non-unit** vectors.

$$\begin{bmatrix}\vec{\mathbf{a}}_1\\ \vec{\mathbf{a}}_2\\ \vec{\mathbf{a}}_3\end{bmatrix} = {}^a\mathbb{R}^b \begin{bmatrix}\vec{\mathbf{b}}_1\\ \vec{\mathbf{b}}_2\\ \vec{\mathbf{b}}_3\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}\vec{\mathbf{b}}_1\\ \vec{\mathbf{b}}_2\\ \vec{\mathbf{b}}_3\end{bmatrix} = {}^b\mathbb{R}^a \begin{bmatrix}\vec{\mathbf{a}}_1\\ \vec{\mathbf{a}}_2\\ \vec{\mathbf{a}}_3\end{bmatrix}$$

Transposing without orthogonality $({}^a\mathbb{R}^b)^T \neq {}^b\mathbb{R}^a$

$$\begin{aligned}[\vec{\mathbf{a}}_1 \;\; \vec{\mathbf{a}}_2 \;\; \vec{\mathbf{a}}_3] &= [\vec{\mathbf{b}}_1 \;\; \vec{\mathbf{b}}_2 \;\; \vec{\mathbf{b}}_3]\,({}^a\mathbb{R}^b)^T \\ [\vec{\mathbf{b}}_1 \;\; \vec{\mathbf{b}}_2 \;\; \vec{\mathbf{b}}_3] &= [\vec{\mathbf{a}}_1 \;\; \vec{\mathbf{a}}_2 \;\; \vec{\mathbf{a}}_3]\,({}^a\mathbb{R}^b)^T\end{aligned} \qquad (6)$$

The definitions of the relationship matrix $\mathbb{R}$ and rotation matrix R follow directly from eqn (4.1). The rotation matrix $\;{}^aR^b \underset{(1)}{\triangleq} \begin{bmatrix}\widehat{\mathbf{a}}_x\\ \widehat{\mathbf{a}}_y\\ \widehat{\mathbf{a}}_z\end{bmatrix} \cdot [\widehat{\mathbf{b}}_x \;\; \widehat{\mathbf{b}}_y \;\; \widehat{\mathbf{b}}_z]$ is a special case of the relationship matrix $\;{}^a\mathbb{R}^b \triangleq \dfrac{1}{\vec{\mathbf{b}}_1 \cdot (\vec{\mathbf{b}}_2 \times \vec{\mathbf{b}}_3)} \begin{bmatrix}\vec{\mathbf{a}}_1\\ \vec{\mathbf{a}}_2\\ \vec{\mathbf{a}}_3\end{bmatrix} \cdot [\vec{\mathbf{b}}_2 \times \vec{\mathbf{b}}_3 \;\; \vec{\mathbf{b}}_3 \times \vec{\mathbf{b}}_1 \;\; \vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2]$.

Optional: Rotation matrices and vector (or dyadic) measures

Given two sets of right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, Section 4.6 proves an arbitrary vector $\vec{\mathbf{v}}$ can be expressed in terms of scalars v_x, v_y, v_z or v_1, v_2, v_3 as shown right [above eqn (7)].

$$\begin{aligned}\vec{\mathbf{v}} &= v_x\,\widehat{\mathbf{a}}_x + v_y\,\widehat{\mathbf{a}}_y + v_z\,\widehat{\mathbf{a}}_z\\ \vec{\mathbf{v}} &= v_1\,\widehat{\mathbf{b}}_x + v_2\,\widehat{\mathbf{b}}_y + v_3\,\widehat{\mathbf{b}}_z\end{aligned}$$

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}_b = {}^bR^a \begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}_a \qquad (7)$$

The scalars v_1, v_2, v_3 are related to v_x, v_y, v_z as shown in eqn (7). The more compact form in eqn (8) relates the 3×1 basis a and basis b column matrix representations of $\vec{\mathbf{v}}$ [eqns (7), (8) are proved in Section 5.8.2].

$$[\vec{\mathbf{v}}]_b = {}^bR^a\,[\vec{\mathbf{v}}]_a \qquad (8)$$

Squash rule for expressing vectors

Similarly, eqn (9) relates the 3×3 matrix representations of a dyadic $\vec{\vec{\mathbf{D}}}$ expressed in the a and b bases (proved in Section 15.3).

$$[\vec{\vec{\mathbf{D}}}]_b = {}^bR^a\,[\vec{\vec{\mathbf{D}}}]_a\,{}^aR^b \qquad (9)$$

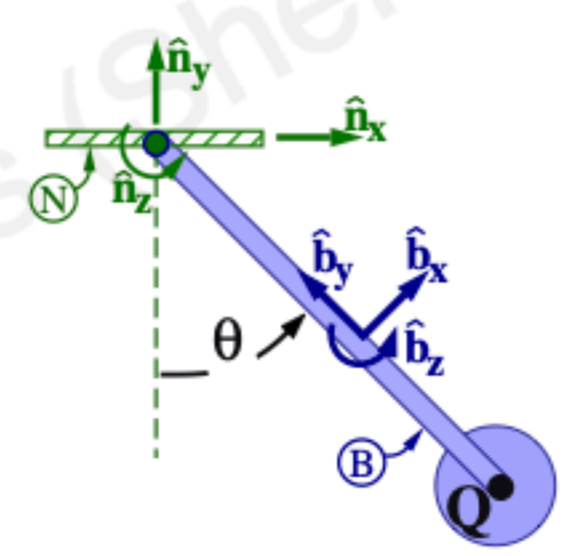
Squash rule for expressing dyadics

Eqn (10) relates the ***skew symmetric matrix representations*** of a vector $\vec{\mathbf{v}}$ expressed in the a and b bases (proved in Section 5.8.3).
Although eqn (10) helps proofs, it is more efficient to use $\text{skew}[\vec{\mathbf{v}}]_b = \text{skew}({}^bR^a\,[\vec{\mathbf{v}}]_a)$.

$$\text{skew}[\vec{\mathbf{v}}]_b = {}^bR^a\,\text{skew}[\vec{\mathbf{v}}]_a\,{}^aR^b \qquad (10)$$

Squash rule for skew-symmetric matrices

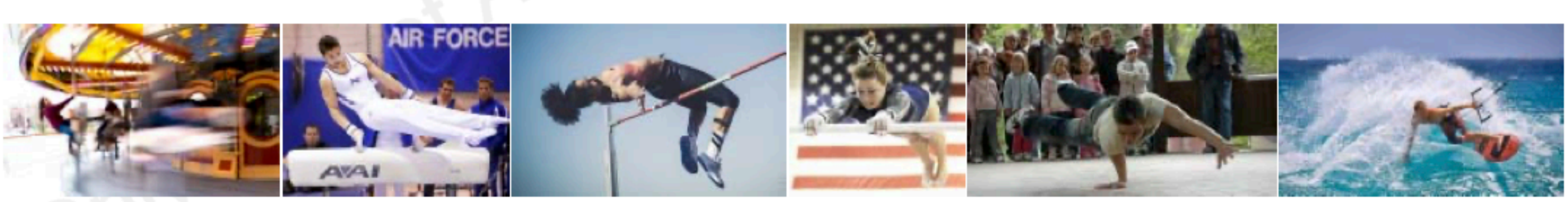
5.3 Rotation matrices - why are they so important?



In 2D (**two dimensional**) analysis, the orientation of a rigid body B in a frame N can be characterized with a **single angle**, e.g., the pendulum angle θ shown right.
Note: Even for this simple 2D pendulum, it is helpful to form the rotation matrix ${}^nR^b$ between $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ as the gravity force on B is $-m\,g\,\widehat{\mathbf{n}}_y$, whereas Q's velocity is $L\,\dot{\theta}\,\widehat{\mathbf{b}}_x$ and Q's acceleration is $L\,\ddot{\theta}\,\widehat{\mathbf{b}}_x - L\,\dot{\theta}^2\,\widehat{\mathbf{b}}_y$. ${}^nR^b$ helps carry out dot-products, e.g., $\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{n}}_y \cdot \widehat{\mathbf{b}}_y$.

In 3D analysis, the orientation of a rigid body B (e.g., a spiraling, wobbling, football) in a reference frame N (e.g., a stadium) <u>cannot</u> be characterized by a single angle.[a] One convenient way to measure B's orientation in N is with a rotation matrix.

[a] **Advanced 3D (Chapter 6).** A rigid body can be oriented with: One angle ***and*** a vector (Section 6.4), or 4 parameters (quaternion) (Section 6.5), or 3 Euler angles (Section 6.2), or ...



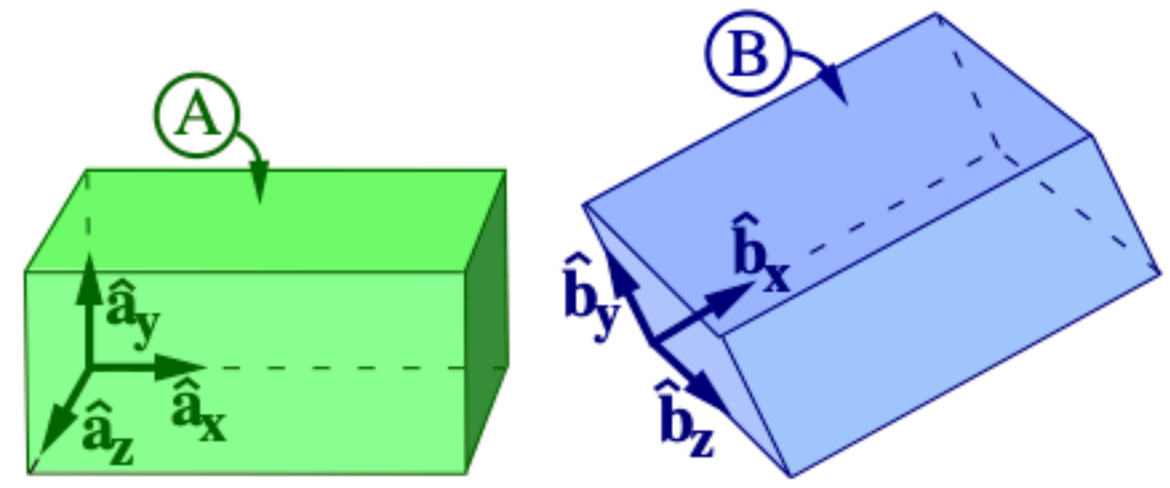
5.4 Examples of how to use a rotation matrix

5.4.1 Example: Calculating angles between unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$

A ***rotation table*** stores ***dot-products*** between unit vectors and makes it easy to calculate angles between unit vectors. For example, ${}^aR^b$ stores $\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_z = 0.2588$ in the $\widehat{\mathbf{a}}_x$ row and $\widehat{\mathbf{b}}_z$ column. Using the definition of dot-product in equation (2.2), the angle between $\widehat{\mathbf{a}}_x$ and $\widehat{\mathbf{b}}_z$ can be calculated as shown below.

${}^aR^b$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_y$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_x$	0.96225	-0.084186	0.2588
$\widehat{\mathbf{a}}_y$	0.17008	0.92840	-0.33037
$\widehat{\mathbf{a}}_z$	-0.21248	0.36192	0.90767

$$\angle(\widehat{\mathbf{a}}_x, \widehat{\mathbf{b}}_z) \;=\; \text{acos}(0.2588) \;\approx\; 75°$$

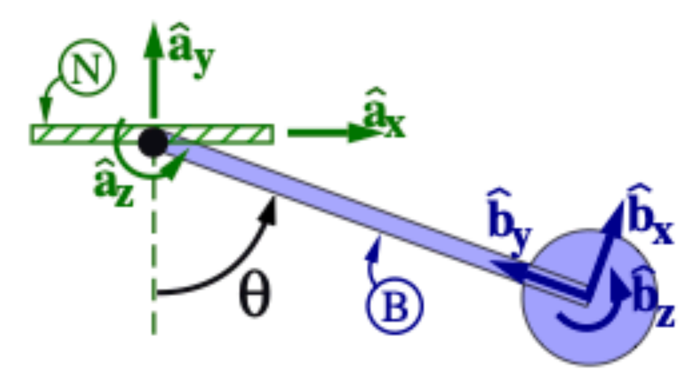


5.4.2 Example: Calculation of rotation matrix inverse (for efficiency, use the transpose).

The following rotation matrix R relates two right-handed, orthogonal, unitary bases.
Since a rotation matrix is orthogonal, its inverse can be written down by inspection (just transpose it).

$$R \;=\; \begin{bmatrix} 0.433 & 0.0580 & 0.8995 \\ -0.25 & 0.9665 & 0.0580 \\ -0.866 & -0.25 & 0.4330 \end{bmatrix} \qquad \Rightarrow \qquad R^{-1} \;=\; \begin{bmatrix} 0.433 & -0.25 & -0.866 \\ 0.0580 & 0.9665 & -0.25 \\ 0.8995 & 0.0580 & 0.4330 \end{bmatrix}$$

5.4.3 Example: Rotation matrix for express, dot, and cross



The figure to the right shows a rod B connected to a fixed support A by a revolute joint. Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, are fixed in A and B. The ${}^bR^a$ rotation table is given below-right.

A rotation table can be used to ***express a vector*** in another basis and perform dot-products and cross-products. For example,

${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

Express First row of ${}^bR^a$ gives: $\widehat{\mathbf{b}}_x \;=\; \cos(\theta)\,\widehat{\mathbf{a}}_x \;+\; \sin(\theta)\,\widehat{\mathbf{a}}_y$

Express First column of ${}^bR^a$ gives: $\widehat{\mathbf{a}}_x \;=\; \cos(\theta)\,\widehat{\mathbf{b}}_x \;-\; \sin(\theta)\,\widehat{\mathbf{b}}_y$

Dot product Element in $\widehat{\mathbf{b}}_y$ row and $\widehat{\mathbf{a}}_x$ column: $\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{a}}_x \;=\; -\sin(\theta)$

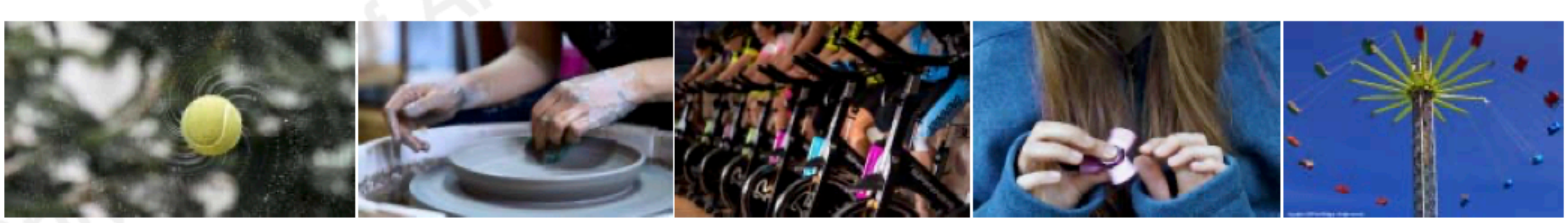
Dot product
$$\begin{aligned}(\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y) \cdot (x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y) \;&=\; x\,(\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_x) \;+\; y\,(\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{b}}_y) \;+\; 2\,x\,(\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{b}}_x) \;+\; 2\,y\,(\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{b}}_y) \\ &=\; x\,[\cos(\theta)] \;+\; y\,[-\sin(\theta)] \;+\; 2\,x\,[\sin(\theta)] \;+\; 2\,y\,[\cos(\theta)]\end{aligned}$$

Examples of mixed-basis **cross-products** (which first *expresses* unit vectors in a consistent basis) are:

$$\widehat{\mathbf{b}}_x \times \big(x\,\widehat{\mathbf{a}}_x + y\,\widehat{\mathbf{a}}_y\big) \;=\; [\cos(\theta)\,\widehat{\mathbf{a}}_x \;+\; \sin(\theta)\,\widehat{\mathbf{a}}_y] \;\times\; (x\,\widehat{\mathbf{a}}_x + y\,\widehat{\mathbf{a}}_y) \;=\; [y\,\cos(\theta) \;-\; x\,\sin(\theta)]\;\widehat{\mathbf{a}}_z$$

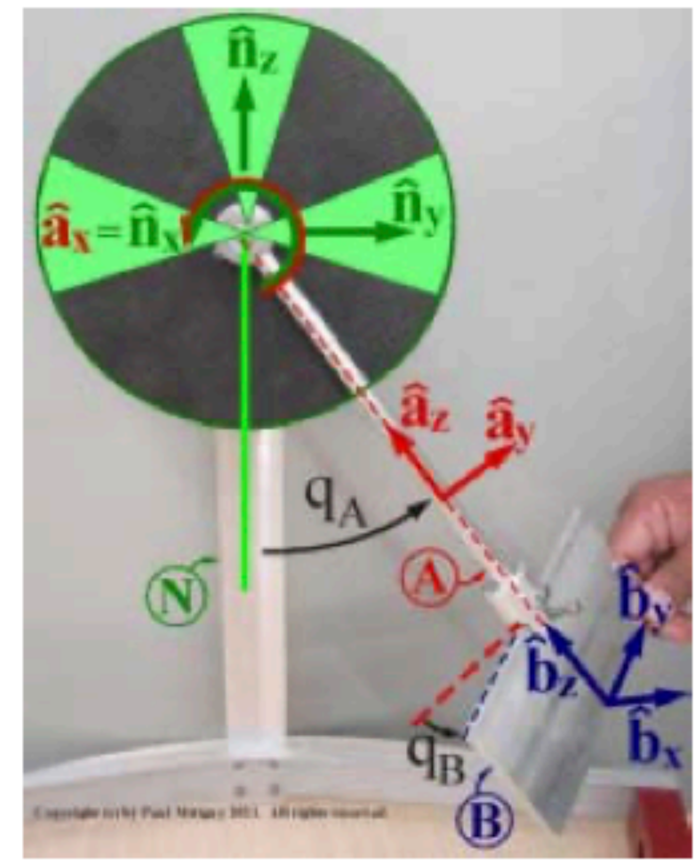
$$\begin{aligned}(\widehat{\mathbf{a}}_x + 2\,\widehat{\mathbf{a}}_y) \times (x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y) \;&=\; \Big\{[\cos(\theta) + 2\,\sin(\theta)]\,\widehat{\mathbf{b}}_x \;+\; [2\,\cos(\theta) - \sin(\theta)]\,\widehat{\mathbf{b}}_y\Big\} \times (x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y) \\ &=\; \{y\,[\cos(\theta) + 2\,\sin(\theta)] \;-\; x\,[2\,\cos(\theta) - \sin(\theta)]\}\;\widehat{\mathbf{b}}_z\end{aligned}$$

Angular velocity: Since $\widehat{\mathbf{b}}_z = \widehat{\mathbf{a}}_z$, B has a simple angular velocity in A of $\boxed{{}^A\vec{\boldsymbol{\omega}}^B \;=\; +\dot{\theta}\,\widehat{\mathbf{b}}_z}$. The $+$ sign results from the ***right-hand rule*** (point the forefingers of your right-hand in the direction of $\widehat{\mathbf{a}}_y$ and curl them towards $\widehat{\mathbf{b}}_y$).



5.5 Simple rotation matrices

Shown to the right is a rigid rod A connected to a fixed support N by a revolute joint whose axis is parallel to the unit vectors $\widehat{\mathbf{n}}_{\mathrm{x}} = \widehat{\mathbf{a}}_{\mathrm{x}}$.
Rod A's orientation in N is characterized by the right-handed rotation $q_A\,\widehat{\mathbf{a}}_{\mathrm{x}}$.



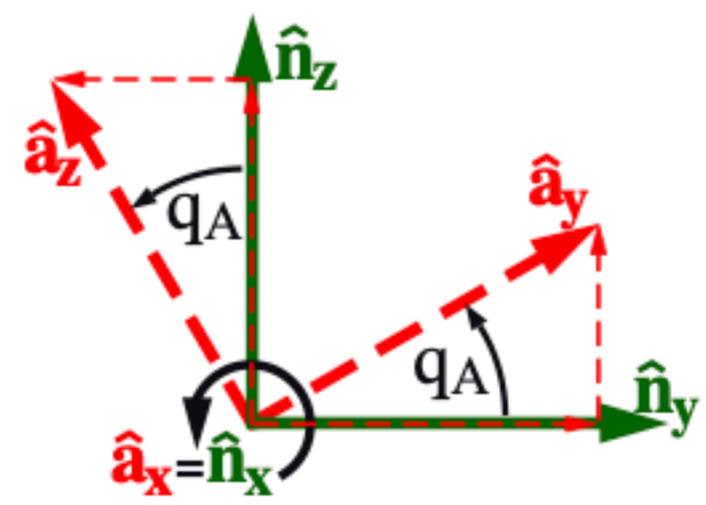
Rigid plate B is connected to rod A by another revolute joint whose axis is parallel to the unit vectors $\widehat{\mathbf{a}}_{\mathrm{z}} = \widehat{\mathbf{b}}_{\mathrm{z}}$ (B can rotate freely about A's long axis).
Plate B's orientation in N is characterized by the right-handed rotation $q_B\,\widehat{\mathbf{b}}_{\mathrm{z}}$.
Note: The B-to-A revolute joint is **perpendicular** to the A-to-N revolute joint.

There are three sets of orthogonal unit vectors, one fixed in each of N, A, B: namely $\widehat{\mathbf{n}}_{\mathrm{x}}, \widehat{\mathbf{n}}_{\mathrm{y}}, \widehat{\mathbf{n}}_{\mathrm{z}}$ and $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ and $\widehat{\mathbf{b}}_{\mathrm{x}}, \widehat{\mathbf{b}}_{\mathrm{y}}, \widehat{\mathbf{b}}_{\mathrm{z}}$.
This problem relates these three sets of unit vectors with rotation matrices.[a]

[a] Simple rotation matrices are formed using the definitions of sine and cosine (SohCahToa).

5.5.1 Example: Forming the simple rotation matrix ${}^aR^n$

The rotation matrix ${}^aR^n$ relating $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ and $\widehat{\mathbf{n}}_{\mathrm{x}}, \widehat{\mathbf{n}}_{\mathrm{y}}, \widehat{\mathbf{n}}_{\mathrm{z}}$ is a ***simple rotation matrix*** because the revolute joint constantly enforces $\widehat{\mathbf{a}}_{\mathrm{x}} = \widehat{\mathbf{n}}_{\mathrm{x}}$.



To form ${}^aR^n$, it is helpful to **draw** $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ and $\widehat{\mathbf{n}}_{\mathrm{x}}, \widehat{\mathbf{n}}_{\mathrm{y}}, \widehat{\mathbf{n}}_{\mathrm{z}}$ in the geometrically suggestive way shown right. After using the definitions of sine and cosine to express each of $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ in terms of $\widehat{\mathbf{n}}_{\mathrm{x}}, \widehat{\mathbf{n}}_{\mathrm{y}}, \widehat{\mathbf{n}}_{\mathrm{z}}$ form the ${}^aR^n$ rotation table as shown below.

$$\begin{aligned}\widehat{\mathbf{a}}_{\mathrm{x}} &= \widehat{\mathbf{n}}_{\mathrm{x}} \\ \widehat{\mathbf{a}}_{\mathrm{y}} &= \cos(q_A)\,\widehat{\mathbf{n}}_{\mathrm{y}} + \sin(q_A)\,\widehat{\mathbf{n}}_{\mathrm{z}} \\ \widehat{\mathbf{a}}_{\mathrm{z}} &= \text{-}\sin(q_A)\,\widehat{\mathbf{n}}_{\mathrm{y}} + \cos(q_A)\,\widehat{\mathbf{n}}_{\mathrm{z}}\end{aligned} \quad\Rightarrow$$

${}^aR^n$	$\widehat{\mathbf{n}}_{\mathrm{x}}$	$\widehat{\mathbf{n}}_{\mathrm{y}}$	$\widehat{\mathbf{n}}_{\mathrm{z}}$
$\widehat{\mathbf{a}}_{\mathrm{x}}$	1	0	0
$\widehat{\mathbf{a}}_{\mathrm{y}}$	0	$\cos(q_A)$	$\sin(q_A)$
$\widehat{\mathbf{a}}_{\mathrm{z}}$	0	$\text{-}\sin(q_A)$	$\cos(q_A)$

Related: Since $\widehat{\mathbf{a}}_{\mathrm{x}} = \widehat{\mathbf{n}}_{\mathrm{x}}$, A has a simple angular velocity in N of $\boxed{{}^N\vec{\boldsymbol{\omega}}^A = +\dot{q}_A\,\widehat{\mathbf{n}}_{\mathrm{x}}}$.
The + sign results from the ***right-hand rule*** (point the forefingers of your right-hand in the direction of $\widehat{\mathbf{n}}_{\mathrm{y}}$ and curl them towards $\widehat{\mathbf{a}}_{\mathrm{y}}$).

The rotation **table** makes it easy to form ${}^aR^n$, the rotation **matrix** relating $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ to $\widehat{\mathbf{n}}_{\mathrm{x}}, \widehat{\mathbf{n}}_{\mathrm{y}}, \widehat{\mathbf{n}}_{\mathrm{z}}$.

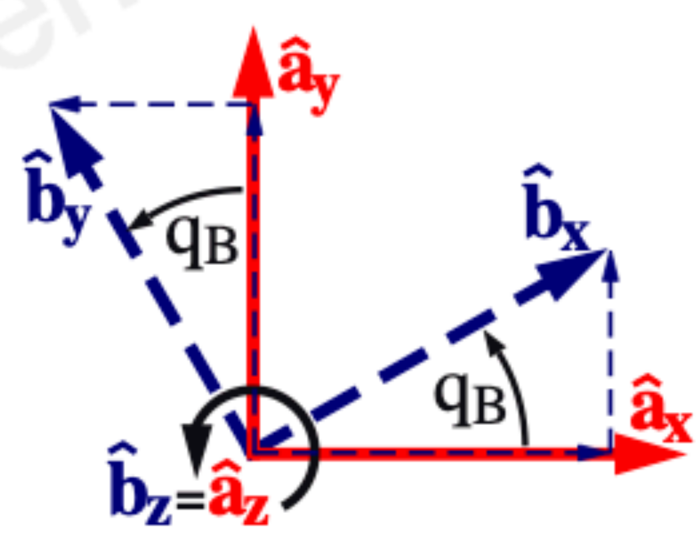
$$\begin{bmatrix}\widehat{\mathbf{a}}_{\mathrm{x}}\\ \widehat{\mathbf{a}}_{\mathrm{y}}\\ \widehat{\mathbf{a}}_{\mathrm{z}}\end{bmatrix} = {}^aR^n \begin{bmatrix}\widehat{\mathbf{n}}_{\mathrm{x}}\\ \widehat{\mathbf{n}}_{\mathrm{y}}\\ \widehat{\mathbf{n}}_{\mathrm{z}}\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}\widehat{\mathbf{a}}_{\mathrm{x}}\\ \widehat{\mathbf{a}}_{\mathrm{y}}\\ \widehat{\mathbf{a}}_{\mathrm{z}}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(q_A) & \sin(q_A)\\ 0 & \text{-}\sin(q_A) & \cos(q_A)\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{n}}_{\mathrm{x}}\\ \widehat{\mathbf{n}}_{\mathrm{y}}\\ \widehat{\mathbf{n}}_{\mathrm{z}}\end{bmatrix}$$

The rotation matrix ${}^nR^a$ is quickly and accurately calculated by the ***transpose*** of ${}^aR^n$.

$$\begin{bmatrix}\widehat{\mathbf{n}}_{\mathrm{x}}\\ \widehat{\mathbf{n}}_{\mathrm{y}}\\ \widehat{\mathbf{n}}_{\mathrm{z}}\end{bmatrix} = {}^nR^a \begin{bmatrix}\widehat{\mathbf{a}}_{\mathrm{x}}\\ \widehat{\mathbf{a}}_{\mathrm{y}}\\ \widehat{\mathbf{a}}_{\mathrm{z}}\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}\widehat{\mathbf{n}}_{\mathrm{x}}\\ \widehat{\mathbf{n}}_{\mathrm{y}}\\ \widehat{\mathbf{n}}_{\mathrm{z}}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(q_A) & \text{-}\sin(q_A)\\ 0 & \sin(q_A) & \cos(q_A)\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{a}}_{\mathrm{x}}\\ \widehat{\mathbf{a}}_{\mathrm{y}}\\ \widehat{\mathbf{a}}_{\mathrm{z}}\end{bmatrix}$$

5.5.2 Example: Forming the simple rotation matrix ${}^bR^a$

The rotation matrix ${}^bR^a$ relating $\widehat{\mathbf{b}}_{\mathrm{x}}, \widehat{\mathbf{b}}_{\mathrm{y}}, \widehat{\mathbf{b}}_{\mathrm{z}}$ and $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ is a ***simple rotation matrix*** because the revolute joint constantly enforces $\widehat{\mathbf{b}}_{\mathrm{z}} = \widehat{\mathbf{a}}_{\mathrm{z}}$.



To form ${}^bR^a$, it is helpful to **draw** $\widehat{\mathbf{b}}_{\mathrm{x}}, \widehat{\mathbf{b}}_{\mathrm{y}}, \widehat{\mathbf{b}}_{\mathrm{z}}$ and $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$ in the geometrically suggestive way shown right. After using the definitions of sine and cosine to express each of $\widehat{\mathbf{b}}_{\mathrm{x}}, \widehat{\mathbf{b}}_{\mathrm{y}}, \widehat{\mathbf{b}}_{\mathrm{z}}$ in terms of $\widehat{\mathbf{a}}_{\mathrm{x}}, \widehat{\mathbf{a}}_{\mathrm{y}}, \widehat{\mathbf{a}}_{\mathrm{z}}$, one can form the ${}^bR^a$ rotation table as shown below.

$$\begin{aligned}\widehat{\mathbf{b}}_{\mathrm{x}} &= \cos(q_B)\,\widehat{\mathbf{a}}_{\mathrm{x}} + \sin(q_B)\,\widehat{\mathbf{a}}_{\mathrm{y}} \\ \widehat{\mathbf{b}}_{\mathrm{y}} &= \text{-}\sin(q_B)\,\widehat{\mathbf{a}}_{\mathrm{x}} + \cos(q_B)\,\widehat{\mathbf{a}}_{\mathrm{y}} \\ \widehat{\mathbf{b}}_{\mathrm{z}} &= \widehat{\mathbf{a}}_{\mathrm{z}}\end{aligned} \quad\Rightarrow$$

${}^bR^a$	$\widehat{\mathbf{a}}_{\mathrm{x}}$	$\widehat{\mathbf{a}}_{\mathrm{y}}$	$\widehat{\mathbf{a}}_{\mathrm{z}}$
$\widehat{\mathbf{b}}_{\mathrm{x}}$	$\cos(q_B)$	$\sin(q_B)$	0
$\widehat{\mathbf{b}}_{\mathrm{y}}$	$\text{-}\sin(q_B)$	$\cos(q_B)$	0
$\widehat{\mathbf{b}}_{\mathrm{z}}$	0	0	1

Related: Since $\widehat{\mathbf{b}}_{\mathrm{z}} = \widehat{\mathbf{a}}_{\mathrm{z}}$, B has a simple angular velocity in A of $\boxed{{}^A\vec{\boldsymbol{\omega}}^B = +\dot{q}_B\,\widehat{\mathbf{n}}_{\mathrm{z}}}$.
The + sign results from the ***right-hand rule*** (point the forefingers of your right-hand in the direction of $\widehat{\mathbf{a}}_{\mathrm{x}}$ and curl them towards $\widehat{\mathbf{b}}_{\mathrm{x}}$).

5.5.3 Hug rule (a pattern for quickly forming simple rotation matrices)

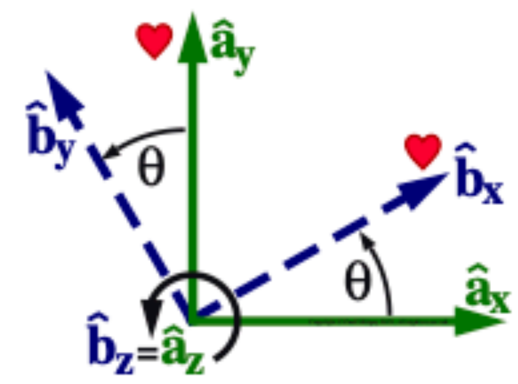
Given: Two sets of orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ initially set with $\widehat{\mathbf{a}}_i = \widehat{\mathbf{b}}_i$ $(i = x, y, z)$, then set $\boldsymbol{b}$ undergoes a simple rotation relative to set $\boldsymbol{a}$ about one of $\widehat{\mathbf{b}}_i = \widehat{\mathbf{a}}_i$ by an angle θ.

Then: The $\widehat{\mathbf{b}}_i$ row and $\widehat{\mathbf{a}}_i$ column of the ${}^bR^a$ rotation table contain only 1 or 0 and the remaining four elements of ${}^bR^a$ have the pattern show right (based on a drawing of the unit vectors with $0 < \theta < 90°$).

$$\begin{matrix} \cos(\theta) & \pm\sin(\theta) \\ \pm\sin(\theta) & \cos(\theta) \end{matrix}$$

The $\pm$ sign is **+** if the unit vector is **hugged** or **−** if "not hugged".
[***Hug rule*** analogy courtesy of Dr. Mandy Koop].

Example: As drawn $\widehat{\mathbf{b}}_x$ is **hugged** $+\sin(\theta)$ since it is between $\widehat{\mathbf{a}}_x$ and $\widehat{\mathbf{a}}_y$, whereas $\widehat{\mathbf{b}}_y$ is not hugged.
Likewise, as drawn $\widehat{\mathbf{a}}_y$ is **hugged** $+\sin(\theta)$ since it is between $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$, whereas $\widehat{\mathbf{a}}_x$ is not hugged.



${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$+\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

5.6 Forming rotation matrices with matrix multiplication

Equation (4) shows the rotation matrix ${}^aR^d$ can be formed by successive matrix multiplication of the ${}^aR^b$, ${}^bR^c$, ${}^cR^d$ rotation matrices. Applying this general technique to ${}^bR^n$ gives ${}^bR^n \underset{(4)}{=} {}^bR^a * {}^aR^n$.

$${}^aR^d \;=\; {}^aR^b \;*\; {}^bR^c \;*\; {}^cR^d$$

Squash rule for multiplying rotation matrices

$${}^bR^n = \underbrace{\begin{bmatrix} \cos(q_B) & \sin(q_B) & 0 \\ -\sin(q_B) & \cos(q_B) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^bR^a} * \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(q_A) & \sin(q_A) \\ 0 & -\sin(q_A) & \cos(q_A) \end{bmatrix}}_{{}^aR^n} = \begin{bmatrix} \cos(q_B) & \sin(q_B)\cos(q_A) & \sin(q_B)\sin(q_A) \\ -\sin(q_B) & \cos(q_B)\cos(q_A) & \cos(q_B)\sin(q_A) \\ 0 & -\sin(q_A) & \cos(q_A) \end{bmatrix}$$

The ${}^bR^n$ rotation **table** (shown right) is copied from its associated rotation **matrix** and succinctly relates $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$. The utility of the rotation table is shown below.

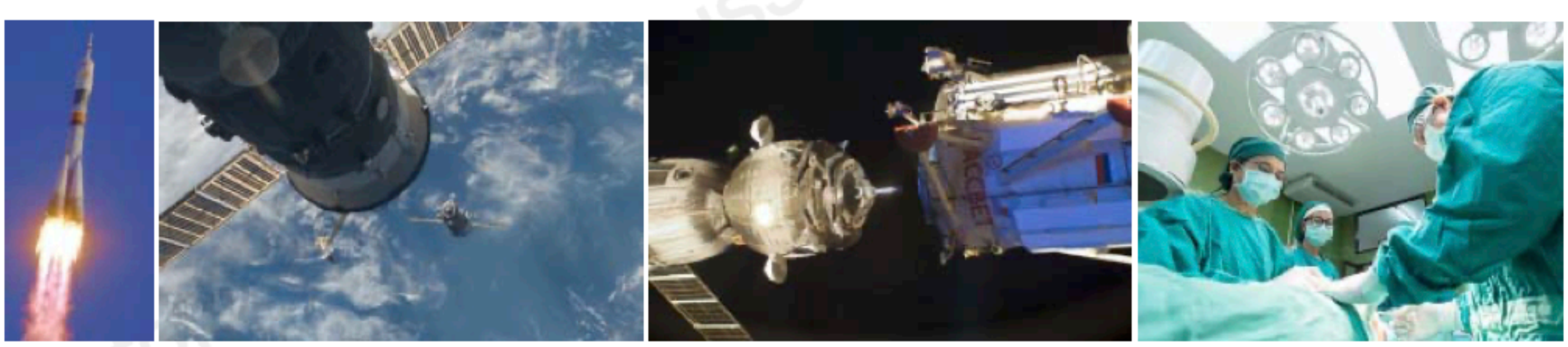
${}^bR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(q_B)$	$\sin(q_B)\cos(q_A)$	$\sin(q_B)\sin(q_A)$
$\widehat{\mathbf{b}}_y$	$-\sin(q_B)$	$\cos(q_B)\cos(q_A)$	$\cos(q_B)\sin(q_A)$
$\widehat{\mathbf{b}}_z$	0	$-\sin(q_A)$	$\cos(q_A)$

Express $\widehat{\mathbf{b}}_x = \cos(q_B)\,\widehat{\mathbf{n}}_x + \sin(q_B)\cos(q_A)\,\widehat{\mathbf{n}}_y + \sin(q_B)\sin(q_A)\,\widehat{\mathbf{n}}_z$ from 1^{st} row of the ${}^bR^n$ rotation table.

Express $\widehat{\mathbf{n}}_x = \cos(q_B)\,\widehat{\mathbf{b}}_x + -\sin(q_B)\,\widehat{\mathbf{b}}_y + 0\,\widehat{\mathbf{b}}_z$ from 1^{st} column of the ${}^bR^n$ rotation table.

$\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_z = \boxed{\sin(q_B)\sin(q_A)}$ element in the $\widehat{\mathbf{b}}_x$ row and $\widehat{\mathbf{n}}_z$ column.

$\angle(\widehat{\mathbf{b}}_x, \widehat{\mathbf{n}}_z) = \boxed{\text{acos}\,[\sin(q_B)\sin(q_A)]}$ angle between $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{n}}_z$ – from the definition $\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_z$.



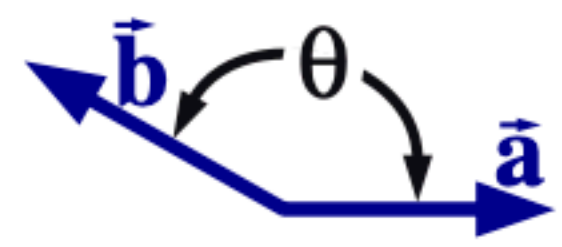
Rotation matrices are used for relative orientation such as the Soyuz spacecraft docking with the international space station, or tele-robotic surgeries with needle/catheter insertion, etc.

5.7 What is an angle?

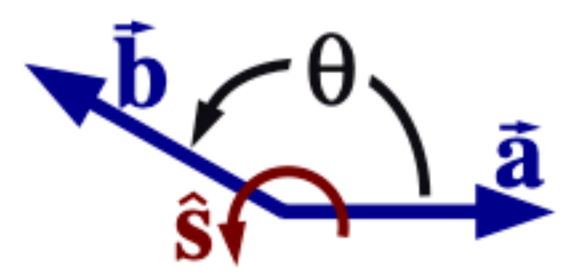
The definition of the word ***"angle"*** is context-dependent, e.g., depending on whether measurements involve sweep between two lines, two vectors (perhaps with a positive sense), two planes, time, etc.[1] [2]

Angle θ *between* two lines $(0 \le \theta \le 90°)$
is defined as the smallest angle between the lines.

Angle θ *between* two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le 180°)$
is defined as the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. It can be calculated as $\theta = \angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \text{acos}(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|})$ (the acos function returns $0 \le \theta \le 180°$).

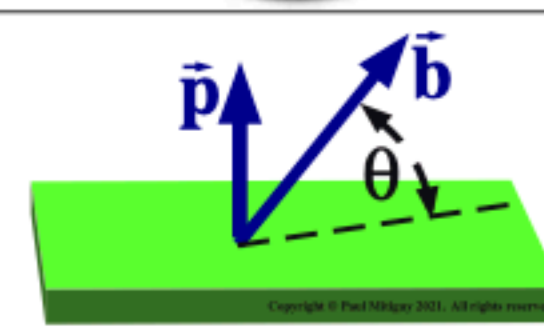


Angle θ *from* vector $\vec{\mathbf{a}}$ *to* vector $\vec{\mathbf{b}}$ with positive sense $\vec{\mathbf{s}}$ $(-180° < \theta \le 180°)$
is positive when $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{s}} > 0$ and negative when $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{s}} < 0$.
It can be calculated as $\theta = \pm\, \text{acos}(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|})$ where $\pm$ is determined by $\text{sign}[(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{s}}]$.

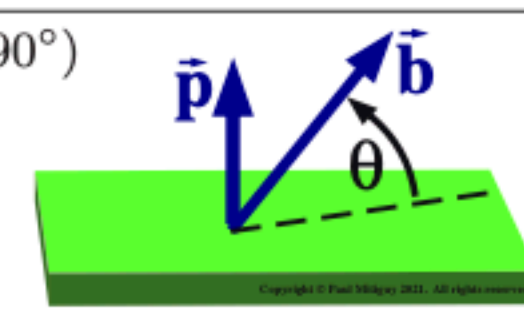


Angle θ *from* vector $\vec{\mathbf{a}}$ *to* vector $\vec{\mathbf{b}}$ with positive sense $\vec{\mathbf{s}}$ and time $(-\infty < \theta < +\infty)$
uses ***"wrap"*** and determines its initial negative/positive sign by $\text{sign}((\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{s}})$.
Although the angle ***between*** two lines or two vectors or two planes is inherently non-negative, an angle can be ***negative*** if there is a positive sense (e.g., clockwise is positive). Shown right, the angle ***between*** the clock's hour-hand and minute-hand is 90°, whereas the angle ***from*** the hour-hand ***to*** the minute-hand with positive clockwise sense is -90°.

Angle θ *between* a plane and a vector $\vec{\mathbf{b}}$ $(0 \le \theta \le 90°)$
is defined as the smallest angle between the plane and vector $\vec{\mathbf{b}}$. It can be calculated $\theta = [90° - \angle(\vec{\mathbf{b}}, \vec{\mathbf{p}})]$, where $\angle(\vec{\mathbf{b}}, \vec{\mathbf{p}})$ is the angle between $\vec{\mathbf{b}}$ and $\vec{\mathbf{p}}$ (perpendicular to the plane). Note there are two perpendicular vectors to a plane (an "inward" and "outward" normal).

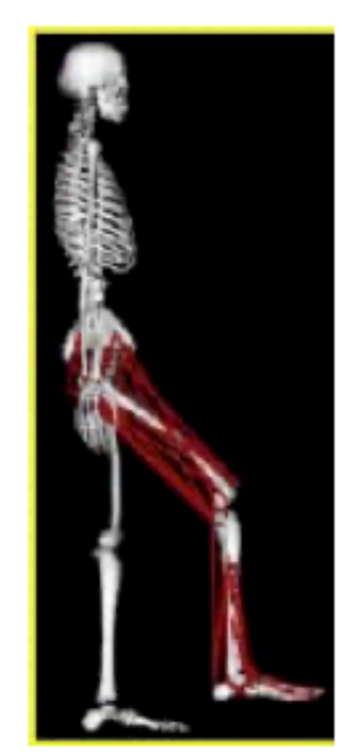


Angle θ *from* a plane *to* a vector with positive plane sense $\vec{\mathbf{p}}$ $(-90° \le \theta \le 90°)$
is well defined if there is one unambiguous vector $\vec{\mathbf{p}}$ perpendicular to the plane. It can be calculated $\theta = \pm[90° - \angle(\vec{\mathbf{b}}, \vec{\mathbf{p}})]$, where $\pm$ is determined by $\text{sign}(\vec{\mathbf{b}} \cdot \vec{\mathbf{p}})$.



θ is sometimes drawn with ⤸ (single arrowhead) if it can be **negative**, otherwise ⤸ (double arrowhead) is used.
Related: Translational measures (e.g., x) are sometimes drawn ⟶ if it can be **negative**, otherwise ⟷ is used.
Note: The figures in the book are being updated to follow this **Hunter/Furman** (SJSU) angle-drawing convention.

In certain applications, angles have names (see table). However, there is no precise universally-agreed definition for these angles, particularly when angles are large. For example, medical doctors and physical therapists have loosely used terms like flexion and extension for centuries. Modern biomechanics rely on accurate measurements of these angles - and require more precise definition (see Section 6.3).



Application	Names of angles
Aerospace and automotive	roll, pitch, yaw or heading, attitude, bank
Gait analysis (hip)	rotation, obliquity, torsion
Knee analysis	flexion/extension, internal/external rotation, adduction/abduction
Surveying and astronomy	inclination/declination, ascension, azimuth, elevation, grade, pitch
Spinning rigid body	nutation, spin, precession, libration, wobble
Diving	somersault, tilt, twist

[1] Height provides a useful analogy to angle. Usually, height is inherently non-negative, e.g., a person's height is a positive quantity. However, one may report height above sea-level as -10 m which implies a **positive upward sense**, hence -10 m is 10 m **below** sea level. Similarly, angles may be negative by providing a positive sense. Historically, angles (e.g., used by the ancient Greeks) predate negative numbers (used by the Europeans ≈ 1500 A.D.) by thousands of years.

[2] For example, the ***"dihedral angle"*** between two planes is the angle between the normals to the two planes.

5.8 Optional: Proofs

5.8.1 Proof that a rotation matrix is orthonormal

${}^aR^b$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_y$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{a}}_x$	R_{xx}	R_{xy}	R_{xz}
$\widehat{\mathbf{a}}_y$	R_{yx}	R_{yy}	R_{yz}
$\widehat{\mathbf{a}}_z$	R_{zx}	R_{zy}	R_{zz}

The ${}^aR^b$ rotation table relates two sets of right-handed orthogonal unit vectors, namely, $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$. Since these are **orthogonal unit** vectors,

$$\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{a}}_i = 1 \quad (i = x, y, z) \qquad \text{and} \qquad \widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{a}}_j = 0 \quad (i \neq j)$$

In view of the ${}^aR^b$ rotation table, $\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{a}}_j$ can also be written in terms of R_{ij} $(i, j, k = x, y, z)$, as

$$\begin{array}{lll}
\widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{a}}_x = 1 = R_{xx}^2 + R_{xy}^2 + R_{xz}^2 & \widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{a}}_y = 0 = R_{xx}\, R_{yx} + R_{xy}\, R_{yy} + R_{xz}\, R_{yz} & \widehat{\mathbf{a}}_x \cdot \widehat{\mathbf{a}}_z = 0 = \ldots \\
\widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{a}}_x = 0 = R_{yx}\, R_{xx} + R_{yy}\, R_{xy} + R_{yz}\, R_{xz} & \widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{a}}_y = 1 = R_{yx}^2 + R_{yy}^2 + R_{yz}^2 & \widehat{\mathbf{a}}_y \cdot \widehat{\mathbf{a}}_z = 0 = \ldots \\
\widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{a}}_x = 0 = \ldots & \widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{a}}_y = 0 = \ldots & \widehat{\mathbf{a}}_z \cdot \widehat{\mathbf{a}}_z = 1 = \ldots
\end{array}$$

To show ${}^aR^b * \left({}^aR^b\right)^T$ equals the identity matrix I, multiply the rotation matrix by its transpose, i.e.,

$$\begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} * \begin{bmatrix} R_{xx} & R_{yx} & R_{zx} \\ R_{xy} & R_{yy} & R_{zy} \\ R_{xz} & R_{yz} & R_{zz} \end{bmatrix} = \begin{bmatrix} R_{xx}^2 + R_{xy}^2 + R_{xz}^2 & R_{xx}\, R_{yx} + R_{xy}\, R_{yy} + R_{xz}\, R_{yz} & \ldots \\ R_{yx}\, R_{xx} + R_{yy}\, R_{xy} + R_{yz}\, R_{xz} & R_{yx}^2 + R_{yy}^2 + R_{yz}^2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

The right-most matrix in the previous equation is the identity matrix I as it elements $\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{a}}_j$ $(i, j, k = x, y, z)$ are 1 or 0. Invoking the definition of the matrix inverse concludes the proof of equation (2), i.e.,

$$\boxed{\left({}^aR^b\right)^{-1} = \left({}^aR^b\right)^T}$$

5.8.2 Proof of relationship between $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ measures of a vector $\vec{\mathbf{v}}$

The proof of equation (7) starts by expressing $\vec{\mathbf{v}}$ in terms of two sets of orthogonal unit vectors namely $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_1$, $\widehat{\mathbf{b}}_2$, $\widehat{\mathbf{b}}_3$ where these are related by the ${}^aR^b$ rotation matrix shown in equation (5).

$$\begin{aligned} \vec{\mathbf{v}} &= v_x\, \widehat{\mathbf{a}}_x + v_y\, \widehat{\mathbf{a}}_y + v_z\, \widehat{\mathbf{a}}_z \\ &= \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{a}}_x \\ \widehat{\mathbf{a}}_y \\ \widehat{\mathbf{a}}_z \end{bmatrix} \underset{(5)}{=} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} {}^aR^b \begin{bmatrix} \widehat{\mathbf{b}}_1 \\ \widehat{\mathbf{b}}_2 \\ \widehat{\mathbf{b}}_3 \end{bmatrix} \end{aligned} \qquad \begin{aligned} \vec{\mathbf{v}} &= v_1\, \widehat{\mathbf{b}}_1 + v_2\, \widehat{\mathbf{b}}_2 + v_3\, \widehat{\mathbf{b}}_3 \\ &= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{b}}_1 \\ \widehat{\mathbf{b}}_2 \\ \widehat{\mathbf{b}}_3 \end{bmatrix} \end{aligned}$$

Equating the previous two expressions, noting $\widehat{\mathbf{b}}_1$, $\widehat{\mathbf{b}}_2$, $\widehat{\mathbf{b}}_3$ are independent (form a 3D basis), and subsequently transposing produces:

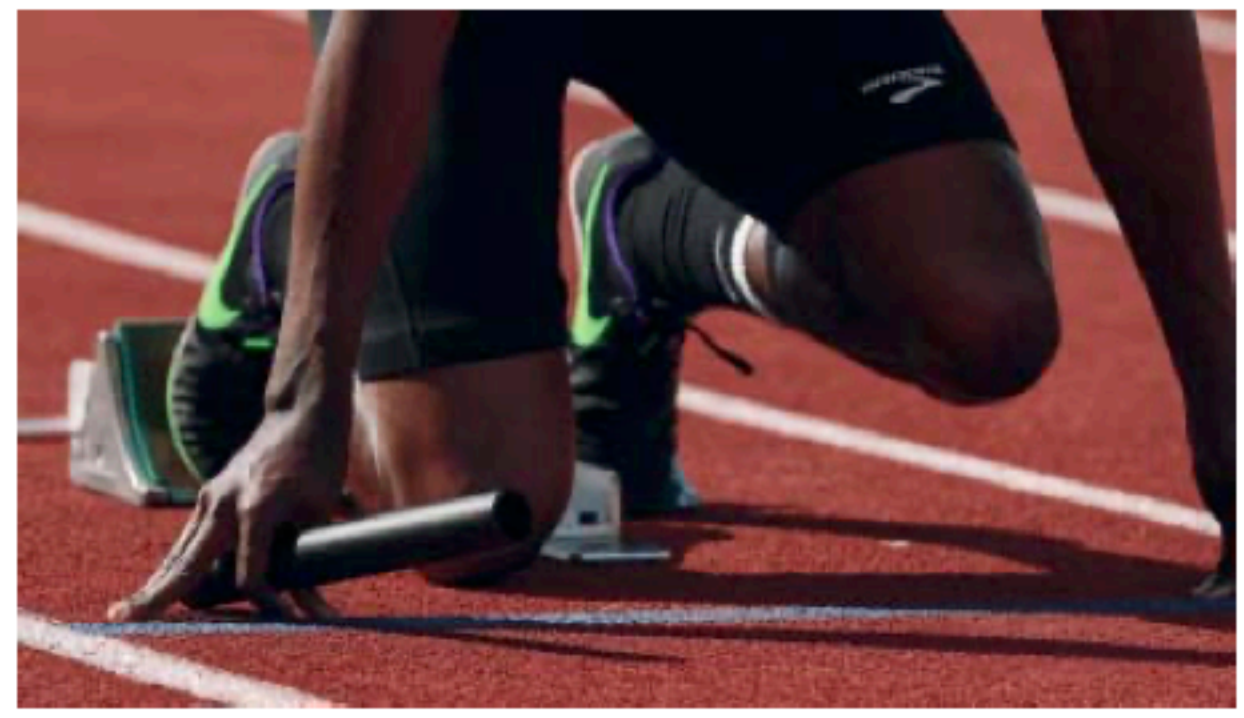
$$\begin{bmatrix} v_x & v_y & v_z \end{bmatrix} {}^aR^b \begin{bmatrix} \widehat{\mathbf{b}}_1 \\ \widehat{\mathbf{b}}_2 \\ \widehat{\mathbf{b}}_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{b}}_1 \\ \widehat{\mathbf{b}}_2 \\ \widehat{\mathbf{b}}_3 \end{bmatrix} \quad \Rightarrow \quad \boxed{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = {}^aR^b \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}$$

5.8.3 Proof of skew-symmetric matrix relationship $\text{skew}[\vec{\mathbf{v}}]_b = {}^bR^a \; \text{skew}[\vec{\mathbf{v}}]_a \; {}^aR^b$

The proof of eqn (10) substitutes $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ from eqn (5) into eqn (2.10) and rearranges as shown below.

$$\text{skew}[\vec{\mathbf{v}}]_b \underset{(2.10)}{=} -\vec{\mathbf{v}} \cdot \underbrace{\begin{bmatrix} \widehat{\mathbf{b}}_x \\ \widehat{\mathbf{b}}_y \\ \widehat{\mathbf{b}}_z \end{bmatrix}}_{{}^bR^a \begin{bmatrix} \widehat{\mathbf{a}}_x \\ \widehat{\mathbf{a}}_y \\ \widehat{\mathbf{a}}_z \end{bmatrix}} \times \underbrace{\begin{bmatrix} \widehat{\mathbf{b}}_x & \widehat{\mathbf{b}}_y & \widehat{\mathbf{b}}_z \end{bmatrix}}_{\begin{bmatrix} \widehat{\mathbf{a}}_x & \widehat{\mathbf{a}}_y & \widehat{\mathbf{a}}_z \end{bmatrix} {}^aR^b} = {}^bR^a \; \underbrace{-\vec{\mathbf{v}} \cdot \begin{bmatrix} \widehat{\mathbf{a}}_x \\ \widehat{\mathbf{a}}_y \\ \widehat{\mathbf{a}}_z \end{bmatrix} \times \begin{bmatrix} \widehat{\mathbf{a}}_x & \widehat{\mathbf{a}}_y & \widehat{\mathbf{a}}_z \end{bmatrix}}_{\text{skew}[\vec{\mathbf{v}}]_a} \; {}^aR^b \underset{(2.10)}{=} {}^bR^a \; \text{skew}[\vec{\mathbf{v}}]_a \; {}^aR^b$$

Chapter 7



Vector differentiation and integration

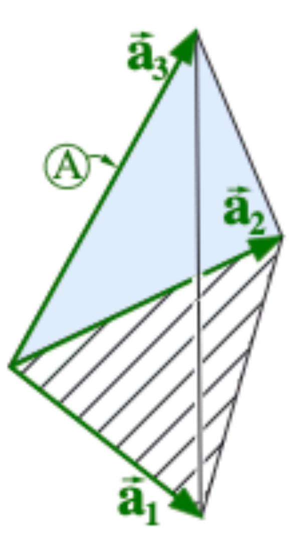
Summary (see examples in Hw 5 and 6)

Vector derivatives play a central role in motion and geometry, e.g., in velocity (time-rate of change of position) and curvature (spatial-rate of change of position). This chapter presents a precise definition for the ***derivative of a vector*** in an arbitrary reference frame A. It also relates vector derivatives in arbitrary reference frames A and B through the ***golden rules for vector differentiation***.

Derivative in reference frame A of $\vec{\mathbf{v}}$.	***Golden rules for vector differentiation (transport theorems)***
$\frac{{}^A d\vec{\mathbf{v}}}{dt} \underset{(1)}{\triangleq} \frac{dv_1}{dt}\vec{\mathbf{a}}_1 + \frac{dv_2}{dt}\vec{\mathbf{a}}_2 + \frac{dv_3}{dt}\vec{\mathbf{a}}_3$	$\frac{{}^A d\vec{\mathbf{v}}}{dt} \underset{(7)}{=} \frac{{}^B d\vec{\mathbf{v}}}{dt} + {}^A\vec{\boldsymbol{\omega}}^B \times \vec{\mathbf{v}}$ $\frac{{}^A d^2\vec{\mathbf{v}}}{dt^2} \underset{(8.12)}{=} \frac{{}^B d^2\vec{\mathbf{v}}}{dt^2} + {}^A\vec{\boldsymbol{\alpha}}^B \times \vec{\mathbf{v}} + {}^A\vec{\boldsymbol{\omega}}^B \times ({}^A\vec{\boldsymbol{\omega}}^B \times \vec{\mathbf{v}}) + 2\,{}^A\vec{\boldsymbol{\omega}}^B \times \frac{{}^B d\vec{\mathbf{v}}}{dt}$
$\frac{{}^A\partial\vec{\mathbf{v}}}{\partial t} \underset{(7.2)}{\triangleq} \frac{\partial v_1}{\partial t}\vec{\mathbf{a}}_1 + \frac{\partial v_2}{\partial t}\vec{\mathbf{a}}_2 + \frac{\partial v_3}{\partial t}\vec{\mathbf{a}}_3$	$\frac{{}^A\partial\vec{\mathbf{v}}}{\partial x} = \frac{{}^B\partial\vec{\mathbf{v}}}{\partial x} + {}^A\vec{\boldsymbol{\omega}}^B_x \times \vec{\mathbf{v}}$ **(Mitiguy)**

A ***reference frame*** is simply a ***rigid object*** such as a rigid body or rigid basis (e.g., orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$).

7.1 Definition: Derivative of a vector in a rigid basis (or reference frame)



Referring to Section 4.6, when $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ form a 3D ***rigid vector basis*** A, ***any*** vector $\vec{\mathbf{v}}$ can be ***expressed*** as shown below in terms of scalars v_1, v_2, v_3. When v_1, v_2, v_3 depend on a ***single*** scalar variable t, one can define the ***derivative in A of $\vec{\mathbf{v}}$ with respect to t***.

$$\vec{\mathbf{v}} \underset{(4.1)}{=} v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3$$

$$\frac{{}^A d\vec{\mathbf{v}}}{dt} \triangleq \frac{dv_1}{dt}\vec{\mathbf{a}}_1 + \frac{dv_2}{dt}\vec{\mathbf{a}}_2 + \frac{dv_3}{dt}\vec{\mathbf{a}}_3 \quad \text{or} \quad \frac{{}^A d\vec{\mathbf{v}}}{dt} \underset{(2)}{\triangleq} \lim_{h\to 0} \frac{\vec{\mathbf{v}}(t+h)|_A - \vec{\mathbf{v}}(t)|_A}{h} \tag{1}$$

More generally, when v_1, v_2, v_3 are functions of the scalar variables s and t, the ***partial derivative in A of $\vec{\mathbf{v}}$ with respect to t*** can be defined as in eqn (2) (also see Section 7.5).

$$\frac{{}^A\partial\vec{\mathbf{v}}}{\partial t} \triangleq \frac{\partial v_1}{\partial t}\vec{\mathbf{a}}_1 + \frac{\partial v_2}{\partial t}\vec{\mathbf{a}}_2 + \frac{\partial v_3}{\partial t}\vec{\mathbf{a}}_3 \quad \text{or} \quad \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t} \underset{(12)}{\triangleq} \lim_{h\to 0} \frac{\vec{\mathbf{v}}(s,t+h)|_A - \vec{\mathbf{v}}(s,t)|_A}{h} \tag{2}$$

The derivative of $\vec{\mathbf{v}}$ in rigid basis $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ is equal to the derivative of $\vec{\mathbf{v}}$ in rigid basis $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ if both bases are fixed in the same reference frame A. The partial derivative of $\vec{\mathbf{v}}$ with respect to the scalar u is independent of A (rigid-basis/reference-frame invariant) <u>if</u> u is unrelated to orientation (e.g., u is mass, a generalized speed, a force/torque measure, or ...).

If u is an orientation variable: (e.g., u is an angle θ or Euler parameter ϵ_0) $\quad \frac{{}^A\partial\vec{\mathbf{v}}}{\partial u} \neq \frac{{}^B\partial\vec{\mathbf{v}}}{\partial u}$	If u is unrelated to orientation: (A and B are any reference frames or rigid bases) $\quad \frac{{}^A\partial\vec{\mathbf{v}}}{\partial u} = \frac{{}^B\partial\vec{\mathbf{v}}}{\partial u} = \frac{\partial\vec{\mathbf{v}}}{\partial u}$

Example: Given a position vector $\vec{\mathbf{r}} = r\,\cos(\theta)\,\widehat{\mathbf{a}}_x + r\,\sin(\theta)\,\widehat{\mathbf{a}}_y + z\,\widehat{\mathbf{a}}_z$, where r, θ, z are time-dependent variables (cylindrical coordinates), then $\frac{{}^A d\vec{\mathbf{r}}}{dt} = [\dot{r}\,\cos(\theta) - r\,\sin(\theta)\,\dot{\theta}]\,\widehat{\mathbf{a}}_x + [\dot{r}\,\sin(\theta) + r\,\cos(\theta)\,\dot{\theta}]\,\widehat{\mathbf{a}}_y + \dot{z}\,\widehat{\mathbf{a}}_z$.

7.2 What is a constant vector (i.e., a vector fixed in a reference frame).

When each of v_1, v_2, v_3 are ***constant***, $\vec{\mathbf{v}}$ is said to be a ***constant vector in A*** ($\vec{\mathbf{v}}$ is ***fixed in A***), and:

- $\vec{\mathbf{v}}$ has a constant magnitude, i.e., $|\vec{\mathbf{v}}| = C$ where C is a constant.
- $\vec{\mathbf{v}}$ has a constant direction in A, i.e., $\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}_i = C_i$ where C_i is a constant and $\vec{\mathbf{a}}_i$ is ***any*** vector **fixed** in A.
- $\frac{^A d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{0}}$ [proved by inspection of equation (1)].

Note: It does not make sense to state that a "vector is constant or fixed" without specifying a reference frame (or rigid basis).
Note: Certain analyses (e.g., ***conservation of translational/angular momentum***) lead to expressions like $\frac{^A d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{0}}$.
Information about $\vec{\mathbf{v}}$ is determined by setting $|\vec{\mathbf{v}}| = C$ or $\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}_i = C_i$.

7.3 Derivative of a constant magnitude vector ($|\vec{\mathbf{v}}|$ is constant)

Concept: $|\vec{\mathbf{v}}|$ is a scalar, so changes in $|\vec{\mathbf{v}}|$ are ***independent of reference frame***.

If $\vec{\mathbf{v}}$ has ***constant magnitude***, $\frac{^F d\vec{\mathbf{v}}}{dt}$ (the derivative of $\vec{\mathbf{v}}$ in <u>any</u> reference frame/rigid basis F) is perpendicular to $\vec{\mathbf{v}}$ [shown in eqn (3)]. This verifies the 2^{nd} example in Section 7.9.

Proof of eqn (3): $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} =$ constant, hence $\frac{d(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})}{dt} = 0$. Section 7.4 gives $\frac{d(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})}{dt} = 2\,\vec{\mathbf{v}} \cdot \frac{^F d\vec{\mathbf{v}}}{dt}$.

If $|\vec{\mathbf{v}}|$ is <u>constant</u>:
$\sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$ is **constant**
$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ is **constant**

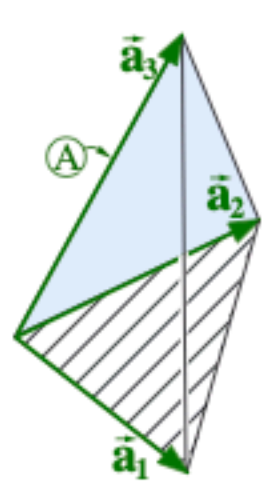
$$\vec{\mathbf{v}} \cdot \frac{^F d\vec{\mathbf{v}}}{dt} = 0 \qquad (3)$$

7.4 Properties of ordinary <u>or</u> partial derivatives of vectors

$$\begin{aligned}
\frac{d(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})}{dt} &= \frac{^A d\,\vec{\mathbf{u}}}{dt} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \frac{^A d\vec{\mathbf{v}}}{dt} & \frac{^A d(s\,\vec{\mathbf{u}})}{dt} &= \frac{ds}{dt}\vec{\mathbf{u}} + s\,\frac{^A d\vec{\mathbf{u}}}{dt} \\
\frac{^A d(\vec{\mathbf{u}} \times \vec{\mathbf{v}})}{dt} &= \frac{^A d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \frac{^A d\vec{\mathbf{v}}}{dt} & \frac{^A d(\vec{\mathbf{u}} + \vec{\mathbf{v}} + \vec{\mathbf{w}})}{dt} &= \frac{^A d\vec{\mathbf{u}}}{dt} + \frac{^A d\vec{\mathbf{v}}}{dt} + \frac{^A d\vec{\mathbf{w}}}{dt} \qquad (4) \\
\frac{^A d(\,\vec{\mathbf{u}} * \vec{\mathbf{v}}\,)}{dt} &= \frac{^A d\vec{\mathbf{u}}}{dt} * \vec{\mathbf{v}} + \vec{\mathbf{u}} * \frac{^A d\vec{\mathbf{v}}}{dt} & \frac{d(\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}})}{dt} &= \frac{^A d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} + \vec{\mathbf{u}} \times \frac{^A d\vec{\mathbf{v}}}{dt} \cdot \vec{\mathbf{w}} + \vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \frac{^A d\vec{\mathbf{w}}}{dt}
\end{aligned}$$

For an independent variable t, a dependent variable $s(t)$, and any reference frame (or rigid basis) A. The 1^{st} property is the ***vector dot-product derivative property*** (the derivative of a scalar does **not** depend on reference frame: proved in Section 7.12). The vector multiplication $\vec{\mathbf{u}} * \vec{\mathbf{v}}$ in the 3^{rd} equation is discussed in Chapter 15 (dyad and dyadics).

7.5 Optional: Expressing a vector in terms of a rigid vector basis



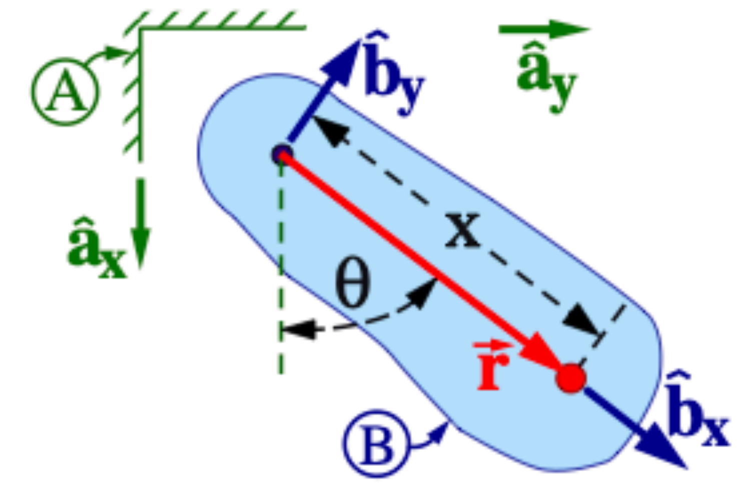
Referring to Section 4.6, when $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ form a 3D ***rigid vector basis*** A, ***any*** vector $\vec{\mathbf{v}}$ can be ***expressed*** as shown below-left in terms of scalars v_1, v_2, v_3.
When one or more of v_1, v_2, v_3 are a function of the scalar variable t, $\vec{\mathbf{v}}$ is a ***vector function of t in A*** and $\vec{\mathbf{v}}$ ***evaluated in A at*** $t = \bar{t}$ is defined as shown eqn (5) (below-right).

$$\vec{\mathbf{v}} \underset{(4.1)}{=} v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3 \qquad \vec{\mathbf{v}}(\bar{t})|_A \triangleq v_1(\bar{t})\,\vec{\mathbf{a}}_1 + v_2(\bar{t})\,\vec{\mathbf{a}}_2 + v_3(\bar{t})\,\vec{\mathbf{a}}_3 \qquad (5)$$

A ***reference frame*** can be constructed by as few as three non-collinear points P_1, P_2, P_3 whose distance from each other are constant. Reference frames and rigid bases are discussed in Section 8.2. One (non-unique) ***rigid vector basis*** that can be constructed from P_1, P_2, P_3 uses: $\vec{\mathbf{a}}_1$ from P_1 to P_2; $\vec{\mathbf{a}}_2$ from P_1 to P_3; and $\vec{\mathbf{a}}_3 = \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2$.

Vector derivatives relate position, velocity, and acceleration.

7.6 Example: Derivative of a vector



A **vector-derivative** is substantially different than a **scalar-derivative** because a vector derivative involves a change in direction (and reference frame) whereas a scalar derivative does not.

To demonstrate how a vector-derivative involves a reference frame, consider a rigid body B that rotates in a vertical plane A. Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in A and B, respectively, with $\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z$ normal to the plane (i.e. $\hat{\mathbf{b}}_z = \hat{\mathbf{b}}_x \times \hat{\mathbf{b}}_y$).

The orientation of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ is determined by first setting $\hat{\mathbf{b}}_i = \hat{\mathbf{a}}_i$ $(i = x, y, z)$ and then B is subjected to a right-handed rotation in A characterized by $\theta\,\hat{\mathbf{a}}_z$ (θ is a variable that depends on time t).

${}^bR^a$	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{\mathbf{b}}_z$	0	0	1

(6)

$${}^A\vec{\boldsymbol{\omega}}^B = \dot{\theta}\,\hat{\mathbf{a}}_z$$

Shown right are calculations of the time-derivatives of $\vec{\mathbf{r}} = x(t)\,\hat{\mathbf{b}}_x$ in B and in A.

$\vec{\mathbf{r}}$'s derivative in **B**	$\vec{\mathbf{r}}$'s derivative in **A**
$\vec{\mathbf{r}} = x\,\hat{\mathbf{b}}_x$	$\vec{\mathbf{r}} \underset{(6)}{=} x\,[\cos(\theta)\,\hat{\mathbf{a}}_x + \sin(\theta)\,\hat{\mathbf{a}}_y]$
$\frac{{}^B d\vec{\mathbf{r}}}{dt} \underset{(1)}{=} \dot{x}\,\hat{\mathbf{b}}_x$	$\frac{{}^A d\vec{\mathbf{r}}}{dt} \underset{(1,6)}{=} \dot{x}\,\underbrace{[\cos(\theta)\,\hat{\mathbf{a}}_x + \sin(\theta)\,\hat{\mathbf{a}}_y]}_{\hat{\mathbf{b}}_x} + x\,\dot{\theta}\,\underbrace{[-\sin(\theta)\,\hat{\mathbf{a}}_x + \cos(\theta)\,\hat{\mathbf{a}}_y]}_{\hat{\mathbf{b}}_y}$

Since $\frac{{}^A d\vec{\mathbf{r}}}{dt} \neq \frac{{}^B d\vec{\mathbf{r}}}{dt}$, it is clear **reference frame makes a difference!**. The ***golden rule for vector differentiation*** relates these derivatives.

$$\frac{{}^A d\vec{\mathbf{r}}}{dt} \underset{(8.1)}{=} \frac{{}^B d\vec{\mathbf{r}}}{dt} + {}^A\vec{\boldsymbol{\omega}}^B \times \vec{\mathbf{r}} \qquad (7)$$

Explained/proved in Chapter 8

Example: $\frac{{}^A d\vec{\mathbf{r}}}{dt} = \frac{{}^B d\vec{\mathbf{r}}}{dt} + {}^A\vec{\boldsymbol{\omega}}^B \times \vec{\mathbf{r}} = \dot{x}\,\hat{\mathbf{b}}_x + \dot{\theta}\,\hat{\mathbf{b}}_z \times x\,\hat{\mathbf{b}}_x = \dot{x}\,\hat{\mathbf{b}}_x + \dot{\theta}\,x\,\hat{\mathbf{b}}_y$.

Since $\vec{\mathbf{r}}$ is a ***vector***, its time-derivative describes its change in ***magnitude*** and ***direction***. However, to determine how $\vec{\mathbf{r}}$'s direction changes, we must ask "with respect to what". For example, $\vec{\mathbf{r}}$'s direction does not change in B since $\vec{\mathbf{r}}$ is always in the $\hat{\mathbf{b}}_x$ direction and $\hat{\mathbf{b}}_x$ is ***fixed*** on B. Conversely, as B rotates in A, $\vec{\mathbf{r}}$'s direction changes in A. The faster B spins in A, the faster $\vec{\mathbf{r}}$'s direction changes in A. You can demonstrate this by spinning in a room while elongating a bike pump. More information is in Chapter 8. This example is reworked in Section 8.3.1.

7.7 Optional: Differential of a vector in a rigid basis (or reference frame)

Referring to Section 7.5, when a vector $\vec{\mathbf{v}}$ is regarded as a function of n independent scalar variables $t_1, \ldots, t_n$ in a rigid basis (or reference frame) A, one may define a quantity ${}^A d\vec{\mathbf{v}}$ in terms of dv_1, dv_2, dv_3 (differentials of scalar variables v_1, v_2, v_3 as described in Section 7.5) or in terms of $dt_1, \ldots, dt_n$ (***differentials of independent variables*** $t_1, \ldots, t_n$). These "independent differentials" are defined as arbitrary (usually small) scalar quantities that have the same dimension of $t_1, \ldots, t_n$.

$$\textbf{(a)}\quad {}^A d\vec{\mathbf{v}} \triangleq dv_1\,\vec{\mathbf{a}}_1 + dv_2\,\vec{\mathbf{a}}_2 + dv_3\,\vec{\mathbf{a}}_3 \quad \text{or} \quad \textbf{(b)}\quad {}^A d\vec{\mathbf{v}} \triangleq \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t_1}\,dt_1 + \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t_2}\,dt_2 + \ldots \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t_n}\,dt_n \qquad (8)$$

${}^A d\vec{\mathbf{v}}$ is called the ***differential in A of*** $\vec{\mathbf{v}}$

When $\vec{\mathbf{v}}$ is regarded as a function of a ***single*** scalar variable t in A, eqn (8.**b**) reduces to the equation shown right. Subsequently dividing both sides by dt gives the ***ratio*** of ${}^A d\vec{\mathbf{v}}$ to dt. Hence, although $\frac{{}^A d\vec{\mathbf{v}}}{dt}$ can always be regarded as a ***ratio of differentials***, it can sometimes be an ***ordinary derivative*** in the sense of eqn (1).

$${}^A d\vec{\mathbf{v}} \underset{(8)}{=} \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t}\,dt$$

$$\frac{{}^A d\vec{\mathbf{v}}}{dt} = \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t}$$

Note: The equivalence of the two definitions for ${}^A d\vec{\mathbf{v}}$ in eqn (8) is shown by substituting $\frac{{}^A\partial\vec{\mathbf{v}}}{\partial t_i} \underset{(2)}{\triangleq} \frac{\partial v_1}{\partial t_i}\vec{\mathbf{a}}_1 + \frac{\partial v_2}{\partial t_i}\vec{\mathbf{a}}_2 + \frac{\partial v_3}{\partial t_i}\vec{\mathbf{a}}_3$ $(i = 1, 2, \ldots, n)$ for ${}^A d\vec{\mathbf{v}}$ in eqn (8.**b**) and factoring on $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$. Next, the coefficient of $\vec{\mathbf{a}}_1$ is seen as the differential of v_1, i.e., $dv_1 \underset{(1.16)}{\triangleq} \frac{\partial f}{\partial t_1}\,dt_1 + \frac{\partial f}{\partial t_2}\,dt_2 + \ldots \frac{\partial f}{\partial t_n}\,dt_n$. Similarly for the coefficients of $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$.

7.8 Optional: Integral of a vector in a rigid basis (or reference frame)

When a vector $\vec{\mathbf{v}}$ is regarded as a function of the scalar variable t in a ***reference frame*** (or ***rigid vector basis***) A, one can define:

Integral in A of $\vec{\mathbf{v}}$

$$
{}^A\!\!\int \vec{\mathbf{v}}\,dt \;\triangleq\; \left(\int v_1\,dt\right)\vec{\mathbf{a}}_1 \;+\; \left(\int v_2\,dt\right)\vec{\mathbf{a}}_2 \;+\; \left(\int v_3\,dt\right)\vec{\mathbf{a}}_3 \qquad (9)
$$

For example, substituting ${}^A d\vec{\mathbf{v}} \;=\; dv_1\,\vec{\mathbf{a}}_1 + dv_2\,\vec{\mathbf{a}}_2 + dv_3\,\vec{\mathbf{a}}_3$ [from eqn (8)] in for $\vec{\mathbf{v}}$ in eqn (7.8) gives

$$
{}^A\!\!\int {}^A d\vec{\mathbf{v}} \underset{(8,\,7.8)}{=} \left(\int dv_1\right)\vec{\mathbf{a}}_1 + \left(\int dv_2\right)\vec{\mathbf{a}}_2 + \left(\int dv_3\right)\vec{\mathbf{a}}_3 \;=\; (v_1+c_1)\,\vec{\mathbf{a}}_1 + (v_2+c_2)\,\vec{\mathbf{a}}_2 + (v_3+c_3)\,\vec{\mathbf{a}}_3
$$

$$
= \;\vec{\mathbf{v}} \;+\; \vec{\mathbf{c}} \qquad \text{where } \vec{\mathbf{c}} \text{ is a constant vector in } A \text{ (i.e., } \vec{\mathbf{c}} \text{ is fixed in } A) \qquad (10)
$$

Eqn (10) is a definition for the ***integral in A of the differential of $\vec{\mathbf{v}}$***.
Section 10.7 shows the utility of eqn (10) for integrating acceleration to find velocity and position.

7.9 Differentation concepts: Changes in magnitude and direction

In **scalar** calculus, $\frac{df}{dt}$ (the ordinary time-derivative of a scalar function f) is defined as shown to the right.

$$
\frac{df}{dt} \;\triangleq\; \lim_{h \to 0} \frac{f(t+h) \;-\; f(t)}{h}
$$

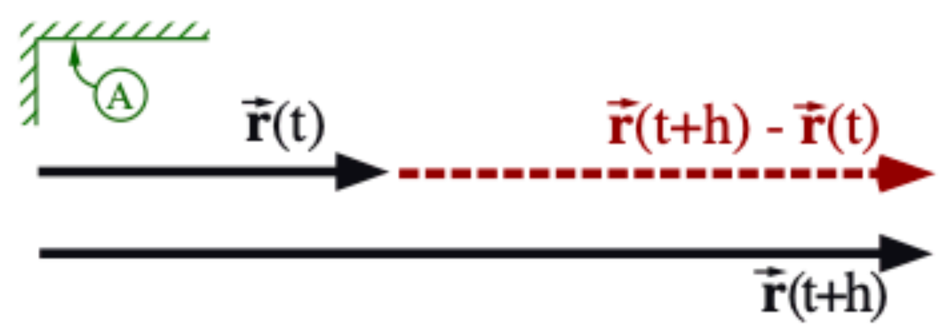
In **vector** calculus, $\frac{{}^A d\vec{\mathbf{v}}}{dt}$ [the ordinary time-derivative in ***reference frame*** A of a vector $\vec{\mathbf{v}}$] is defined as shown right where $\vec{\mathbf{v}}(t+h)|_A$ and $\vec{\mathbf{v}}(t)|_A$ denote $\vec{\mathbf{v}}$ evaluated in A at $t+h$ and t, respectively.

$$
\frac{{}^A d\vec{\mathbf{v}}}{dt} \;\triangleq\; \lim_{h \to 0} \frac{\vec{\mathbf{v}}(t+h)|_A \;-\; \vec{\mathbf{v}}(t)|_A}{h}
$$

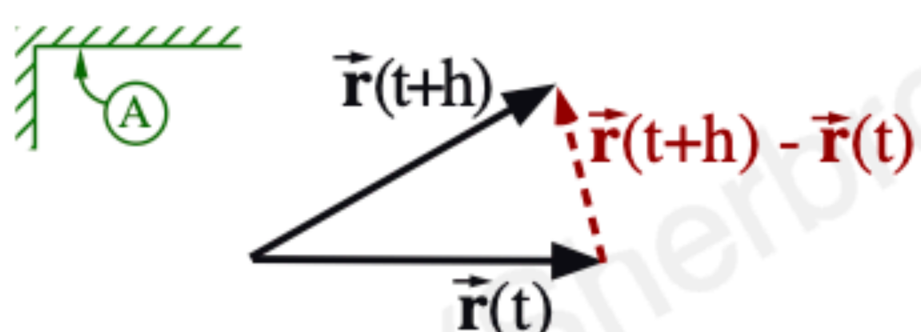
A is a ***reference frame*** or ***rigid vector basis***

Differentiating a **vector** is more complicated than differentiating a scalar because a vector's magnitude can change, its direction in reference frame A can change, or both can change.

For example, the figure to the right shows a vector $\vec{\mathbf{r}}$ whose **magnitude changes** but whose direction in reference frame A remains constant . The vector $\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)$ measures the change in reference frame A of the vector $\vec{\mathbf{r}}$ from time t to time $t+h$. In the limit as $h \to 0$, the direction of $\frac{{}^A d\vec{\mathbf{r}}}{dt}$ is **parallel** to $\vec{\mathbf{r}}$.



Shown right is a vector $\vec{\mathbf{r}}$ whose **magnitude is constant** but whose direction in frame A changes. The vector $\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)$ measures the change in reference frame A of the vector $\vec{\mathbf{r}}$ from time t to time $t+h$. In the limit as $h \to 0$, $\frac{{}^A d\vec{\mathbf{r}}}{dt}$ is **perpendicular** to $\vec{\mathbf{r}}(t)$.



The proof of these statements along with related important concepts are in Section 7.13.

7.10 Optional: Limit of a vector in a reference frame

Referring to Section 7.7, when a vector $\vec{\mathbf{v}}$ is regarded as a function of the scalar variable t in a ***reference frame*** (or ***rigid vector basis***) A, the ***integral in A of $\vec{\mathbf{v}}$*** can be defined as

$$
\lim_{t \to \bar{t}} \vec{\mathbf{v}}(s,t)|_A \;\triangleq\; \left[\lim_{t \to \bar{t}} v_1(s,t)\right]\vec{\mathbf{a}}_1 \;+\; \left[\lim_{t \to \bar{t}} v_2(s,t)\right]\vec{\mathbf{a}}_2 \;+\; \left[\lim_{t \to \bar{t}} v_3(s,t)\right]\vec{\mathbf{a}}_3 \qquad (11)
$$

To connect vector limits with vector differentiation, the limit definition in eqn (11) is applied as shown below. Next, definition (1.15) is used for the ***partial derivatives*** of scalars v_i (i=1, 2, 3) with respect to t. Lastly, the definition in eqn (2) is employed to prove how vector limits relate to vector differentiation.

$$
\lim_{h \to 0} \frac{\vec{\mathbf{v}}(s,t+h)|_A - \vec{\mathbf{v}}(s,t)|_A}{h} \underset{(11)}{\triangleq} \left[\lim_{h \to 0} \frac{v_1(s,t+h) - v_1(s,t)}{h}\right]\vec{\mathbf{a}}_1 + \left[\lim_{h \to 0} \frac{v_2(s,t+h) - v_2(s,t)}{h}\right]\vec{\mathbf{a}}_2 + \left[\lim_{h \to 0} \frac{v_3(s,t+h) - v_3(s,t)}{h}\right]\vec{\mathbf{a}}_3
$$

$$
\underset{(1.15)}{=} \frac{\partial v_1}{\partial t}\,\vec{\mathbf{a}}_1 \;+\; \frac{\partial v_2}{\partial t}\,\vec{\mathbf{a}}_2 \;+\; \frac{\partial v_3}{\partial t}\,\vec{\mathbf{a}}_3 \;\underset{(2)}{=}\; \frac{{}^A\partial\vec{\mathbf{v}}}{\partial t} \qquad (12)
$$

7.11 Optional: Derivative of a scalar with respect to a vector (gradients)

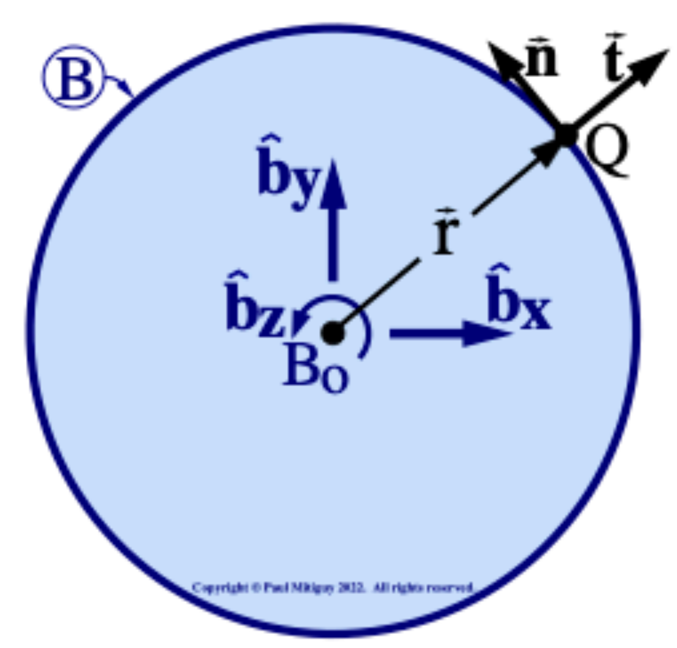
If a **scalar function** F (such as temperature) depends on a **vector** $\vec{\mathbf{v}}$ (such as a position vector), it is useful to define a vector $\vec{\nabla}_{\vec{v}} F$ as shown right in eqn (13) where $\widehat{\mathbf{a}}_x, \widehat{\mathbf{a}}_y, \widehat{\mathbf{a}}_z$ are ***any*** orthogonal unit vectors and $v_i \triangleq \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_i \; (i = x, y, z)$.

$$\vec{\nabla}_{\vec{v}} F \;\triangleq\; \frac{\partial F}{\partial v_x}\,\widehat{\mathbf{a}}_x + \frac{\partial F}{\partial v_y}\,\widehat{\mathbf{a}}_y + \frac{\partial F}{\partial v_z}\,\widehat{\mathbf{a}}_z \qquad (13)$$

$\vec{\nabla}_{\vec{v}} F$ is the ***derivative of F with respect to*** $\vec{\mathbf{v}}$.

$\vec{\nabla}_{\vec{v}} F$ is invariant with respect to the choice of basis vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ (proved [44, pg. 125]). When $\vec{\nabla}_{\vec{v}} F$ produces a force vector, the scalar function F is called a ***force function***. Equation (25.12) relates the ***gradient*** of potential energy to force. When $\vec{\mathbf{v}}$ is a ***position vector***, $\vec{\nabla}_{\vec{v}} F$ is called a ***spatial gradient*** (and is frequently denoted without the subscript as $\vec{\nabla} F$). If F is a continuous function that describes the ***surface of an object***, $\vec{\nabla}_{\vec{v}} F$ is normal to the surface of the object.

7.11.1 Differential geometry: Normal and tangent to a circle



The figure to the right shows a point Q on the periphery of a circle of radius r that is centered at point B_o. Q's position from B_o can be expressed as ${}^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y$ where x and y are scalars and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are right-handed orthogonal unit vectors with $\widehat{\mathbf{b}}_z$ perpendicular to the plane of the circle.

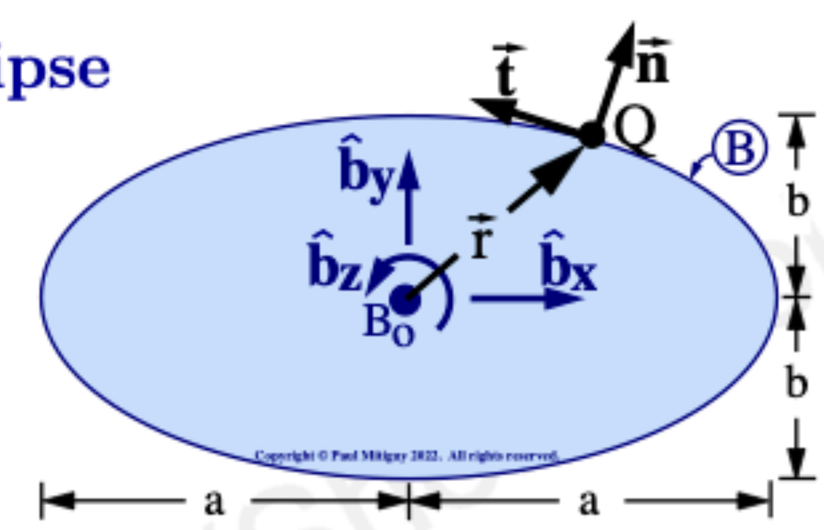
This ***circle*** is defined as the locus of points in the $\widehat{\mathbf{b}}_z$ plane that are a distance r from point B_o. Mathematically, this circles's boundary is described by $\left|{}^{B_o}\vec{\mathbf{r}}^{\,Q}\right| = r$, which gives the scalar function F (shown below) that relates x, y, r.

In general, when a scalar function F describes an object's boundary, the ***spatial gradient*** $\vec{\nabla} F$ is normal to the boundary. As shown right, $\vec{\nabla} F$ calculates an outward normal vector $\vec{\mathbf{n}}$ at point Q and facilitates calculation of a tangent vector $\vec{\mathbf{t}}$ at Q (**directed as shown in the figure**).

$${}^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y \qquad F = x^2 + y^2 - r^2 = 0$$

$$\vec{\nabla} F \underset{(7.13)}{=} \frac{\partial F}{\partial x}\,\widehat{\mathbf{b}}_x + \frac{\partial F}{\partial y}\,\widehat{\mathbf{b}}_y$$
$$\vec{\mathbf{n}} = \vec{\nabla} F = 2\,x\,\widehat{\mathbf{b}}_x + 2\,y\,\widehat{\mathbf{b}}_y$$
$$\vec{\mathbf{t}} = \widehat{\mathbf{b}}_z \times \vec{\mathbf{n}} = -2\,y\,\widehat{\mathbf{b}}_x + 2\,x\,\widehat{\mathbf{b}}_y$$

7.11.2 Differential geometry: Normal and tangent to an ellipse



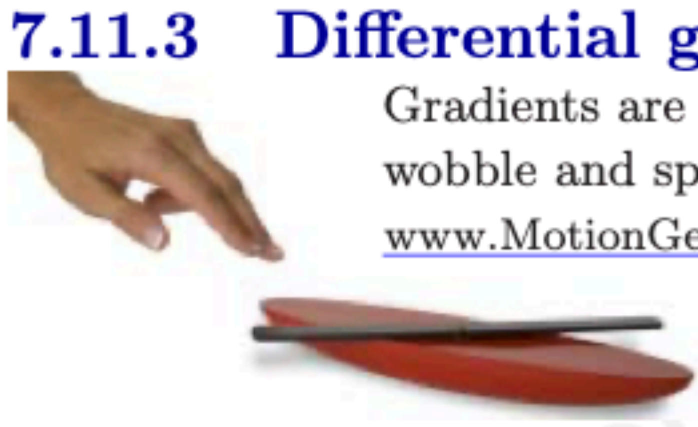
The figure to the right shows a point Q on the periphery of an ellipse of semi-diameters a and b that is centered at point B_o. Q's position from B_o is ${}^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y$ where x and y are scalars and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are right-handed orthogonal unit vectors with $\widehat{\mathbf{b}}_z$ perpendicular to the plane of the ellipse, $\widehat{\mathbf{b}}_x$ along the ellipse's major axis, and $\widehat{\mathbf{b}}_y$ along its minor axis.

In general, when a scalar function F describes an object's boundary, the ***spatial gradient*** $\vec{\nabla} F$ is normal to the boundary. As shown right, $\vec{\nabla} F$ calculates an outward normal vector $\vec{\mathbf{n}}$ at point Q and facilitates calculation of a tangent vector $\vec{\mathbf{t}}$ at Q (**directed as shown in the figure**).

$${}^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y \qquad F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

$$\vec{\nabla} F \underset{(7.13)}{=} \frac{\partial F}{\partial x}\,\widehat{\mathbf{b}}_x + \frac{\partial F}{\partial y}\,\widehat{\mathbf{b}}_y$$
$$\vec{\mathbf{n}} = \vec{\nabla} F = \frac{2\,x}{a^2}\,\widehat{\mathbf{b}}_x + \frac{2\,y}{b^2}\,\widehat{\mathbf{b}}_y$$
$$\vec{\mathbf{t}} = \widehat{\mathbf{b}}_z \times \vec{\mathbf{n}} = \frac{-2\,y}{b^2}\,\widehat{\mathbf{b}}_x + \frac{2\,x}{a^2}\,\widehat{\mathbf{b}}_y$$

7.11.3 Differential geometry: Normal (gradient) to an ellipsoid



Gradients are used for analyzing the non-intuitive wobble and spin reversal of a dynamic celt.
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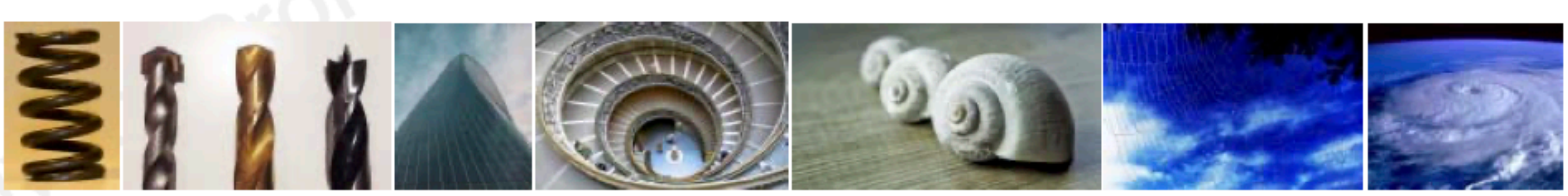
Ellipsoid: $F = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$$\vec{\mathbf{n}} = \vec{\nabla} F \underset{(13)}{=} \frac{\partial F}{\partial x}\,\widehat{\mathbf{b}}_x + \frac{\partial F}{\partial y}\,\widehat{\mathbf{b}}_y + \frac{\partial F}{\partial z}\,\widehat{\mathbf{b}}_z = \frac{2\,x}{a^2}\,\widehat{\mathbf{b}}_x + \frac{2\,y}{b^2}\,\widehat{\mathbf{b}}_y + \frac{2\,z}{c^2}\,\widehat{\mathbf{b}}_z$$



7.12 Optional: Proof of vector dot-product derivative property

One way to prove the ***vector dot-product derivative property*** (first equation in Section 7.4) starts by expressing arbitrary vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in terms of a set of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ fixed in a reference frame (or rigid basis) A and calculating their dot-product.

$$\begin{aligned} \vec{\mathbf{u}} &= u_x\,\widehat{\mathbf{a}}_x + u_y\,\widehat{\mathbf{a}}_y + u_z\,\widehat{\mathbf{a}}_z \\ \vec{\mathbf{v}} &= v_x\,\widehat{\mathbf{a}}_x + v_y\,\widehat{\mathbf{a}}_y + v_z\,\widehat{\mathbf{a}}_z \\ \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= u_x\,v_x + u_y\,v_y + u_z\,v_z \end{aligned} \tag{14}$$

Differentiating the scalar quantity $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$ with respect to the scalar variable t produces

$$\frac{d\,(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{dt} \underset{(14)}{=} \dot{u}_x\,v_x + u_x\,\dot{v}_x + \dot{u}_y\,v_y + u_y\,\dot{v}_y + \dot{u}_z\,v_z + u_z\,\dot{v}_z \tag{15}$$

The next step is to form the right-hand side of the equation being proved, i.e., $\frac{{}^A d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot\frac{{}^A d\vec{\mathbf{v}}}{dt}$, as

$$\begin{aligned} \frac{{}^A d\vec{\mathbf{u}}}{dt} &\underset{(1,14)}{\triangleq} \dot{u}_x\,\widehat{\mathbf{a}}_x + \dot{u}_y\,\widehat{\mathbf{a}}_y + \dot{u}_z\,\widehat{\mathbf{a}}_z & \qquad \frac{{}^A d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} &\underset{(14)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z \\ \frac{{}^A d\vec{\mathbf{v}}}{dt} &\underset{(1,14)}{\triangleq} \dot{v}_x\,\widehat{\mathbf{a}}_x + \dot{v}_y\,\widehat{\mathbf{a}}_y + \dot{v}_z\,\widehat{\mathbf{a}}_z & \qquad \vec{\mathbf{u}}\cdot\frac{{}^A d\vec{\mathbf{v}}}{dt} &\underset{(14)}{=} u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \end{aligned} \tag{16}$$

Combining the two right-most equations in the previous set of equations gives

$$\frac{{}^A d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot\frac{{}^A d\vec{\mathbf{v}}}{dt} \underset{(16)}{=} \dot{u}_x\,v_x + \dot{u}_y\,v_y + \dot{u}_z\,v_z + u_x\,\dot{v}_x + u_y\,\dot{v}_y + u_z\,\dot{v}_z \tag{17}$$

This proof concludes by seeing the equivalence of eqns (15) and (17), and recalling that $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ are fixed in an arbitrary reference frame (or rigid basis) A. Hence, for arbitrary reference frames (or rigid bases) A or B,

$$\boxed{\frac{d(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})}{dt} = \frac{{}^A d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot\frac{{}^A d\vec{\mathbf{v}}}{dt} = \frac{{}^B d\vec{\mathbf{u}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot\frac{{}^B d\vec{\mathbf{v}}}{dt}}$$

Note: The ***vector dot-product derivative property*** is used to prove the uniqueness and existence of the ***golden rule for vector differentiation*** [equation (8.1)] in Section 8.5.1.

This proof does not require (but is greatly simplified with) orthogonal unit vectors. To prove with non-orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, write $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$ in terms of $\widehat{\mathbf{a}}_1\cdot\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_1\cdot\widehat{\mathbf{a}}_3$, $\widehat{\mathbf{a}}_2\cdot\widehat{\mathbf{a}}_3$ (which are the cosines of angles between the unit vectors) and then note that for a **rigid basis** A, the angles between these unit vectors are constant - and hence $\frac{d\,(\widehat{\mathbf{a}}_i\cdot\widehat{\mathbf{a}}_j)}{dt} = 0 \;\; (i,j=1,2,3)$.

7.13 Optional: Proof magnitude-direction properties for vector derivative

Any non-zero vector $\vec{\mathbf{v}}$ can be written in magnitude-direction form as $\vec{\mathbf{v}} = v\,\widehat{\mathbf{u}}$ ($\vec{\mathbf{v}}$'s magnitude v is a positive scalar and $\widehat{\mathbf{u}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$). $\vec{\mathbf{v}}$'s derivative in an arbitrary reference frame A is shown right.

$$\begin{aligned} \vec{\mathbf{v}} &= v\,\widehat{\mathbf{u}} \\ \frac{{}^A d\vec{\mathbf{v}}}{dt} &\underset{(4)}{=} \frac{dv}{dt}\,\widehat{\mathbf{u}} + v\,\frac{{}^A d\widehat{\mathbf{u}}}{dt} \end{aligned} \tag{18}$$

Dot-multiplying eqn (18) first with $\widehat{\mathbf{u}}$ then $\frac{{}^A d\widehat{\mathbf{u}}}{dt}$ produces

$$\begin{aligned} \frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\widehat{\mathbf{u}} &\underset{(18)}{=} \frac{dv}{dt}\,\overbrace{\widehat{\mathbf{u}}\cdot\widehat{\mathbf{u}}}^{1} + v\,\overbrace{\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\widehat{\mathbf{u}}}^{0} = \frac{dv}{dt} \\ \frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} &\underset{(18)}{=} \frac{dv}{dt}\,\overbrace{\widehat{\mathbf{u}}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt}}^{0} + v\,\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} = v\,\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} \end{aligned} \;\Rightarrow\; \boxed{\begin{aligned} \frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\widehat{\mathbf{u}} &= \frac{dv}{dt} \quad \text{(change in } \vec{\mathbf{v}}\text{'s magnitude)} \\ \frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} &= v\,\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} \quad \text{measures } \frac{{}^A d\widehat{\mathbf{u}}}{dt} \end{aligned}} \tag{19}$$

The previous proof uses the fact [from eqn (3)] that for any constant-length vector (e.g., the unit vector $\widehat{\mathbf{u}}$), $\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\widehat{\mathbf{u}} = 0$.

$\frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\widehat{\mathbf{u}} = \frac{dv}{dt}$	is the change in $\vec{\mathbf{v}}$'s magnitude. If $\frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\widehat{\mathbf{u}} = 0$, $\vec{\mathbf{v}}$'s magnitude is **constant**.
$\frac{{}^A d\vec{\mathbf{v}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt} = v\,\frac{{}^A d\widehat{\mathbf{u}}}{dt}\cdot\frac{{}^A d\widehat{\mathbf{u}}}{dt}$	If this is 0, $\frac{{}^A d\widehat{\mathbf{u}}}{dt} = \vec{\mathbf{0}}$ so $\widehat{\mathbf{u}}$'s direction (and hence $\vec{\mathbf{v}}$'s direction) is **constant in A**.