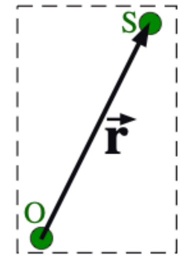
3.1 Position concepts - what objects have a position vector? (Section 3.1)

Draw $\vec{\mathbf{r}}$, the position vector from a point O to some object S.
In general and **without ambiguity**, S could be a (circle all appropriate objects):

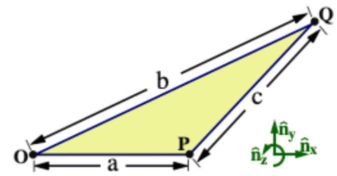


Real number	Line	Set of points	**Center of a circle**
Vector	Triangle	Reference frame	**Mass center of set of particles**
Matrix	**Point**	Rigid body	**Mass center of a rigid body**
3D orthogonal basis	**Particle**	Flexible body	System of particles and bodies

3.2 Calculate the area of a triangle (position-vectors and/or trigonometry)

Length of side	a	2 m
Length of side	b	4 m
Length of side	c	2.4 m
Area of triangle		1.824 m^2

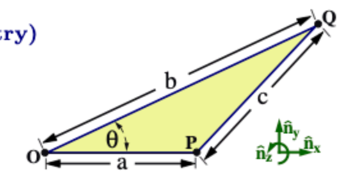
There are multiple ways to do this problem including vectors and trigonometry.
Optional: Calculate area using just $+ \; - \; * \; / \; \sqrt{\cdot}$.
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.



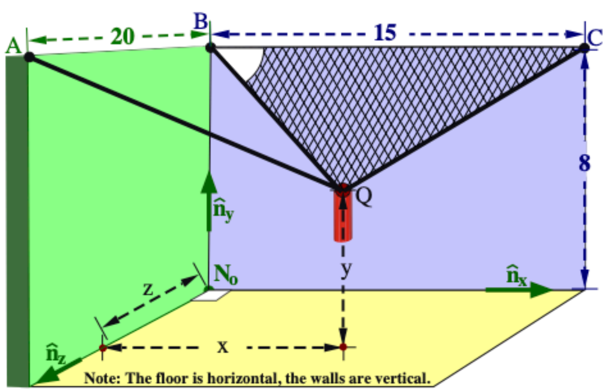
3.3 Calculate the length of a triangle's side (position-vectors or trigonometry)

Length of side	a	2.0 m
Length of side	c	2.4 m
Angle opposite c	θ	25°
Length of side	b	4.059 m

Calculate the length of side b to 3^+ digits.
There are multiple ways to do this problem including vectors and trigonometry.
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.



3.4 Trigonometry: Lengths, angles, and surface area/normal. (Sections 2.9, 2.10, 3.4)



Three cables attach a microphone Q to pegs A, B, C.
Known: Peg and microphone locations from point N_o

Quantity	Symbol	Value
Distance from A to B	AB	20 m
Distance from B to C	BC	15 m
Distance from N_o to B	h	8 m
$\hat{\mathbf{n}}_x$ measure of $^{N_o}\vec{\mathbf{r}}^{Q}$	x	7 m
$\hat{\mathbf{n}}_y$ measure of $^{N_o}\vec{\mathbf{r}}^{Q}$	y	5 m
$\hat{\mathbf{n}}_z$ measure of $^{N_o}\vec{\mathbf{r}}^{Q}$	z	8 m

$$^{N_o}\vec{\mathbf{r}}^{Q} = x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z$$

Calculate/verify (2^+ significant digits) each value in the table to the right.

Note: Δ_{BCQ} is the triangle formed by points B, C, and Q.

Quantity	Symbol	Value
Length of cable joining A and Q	L_A	14.2 m
Length of cable joining B and Q	L_B	11.0 m
Length of cable joining C and Q	L_C	11.7 m
Angle between line $\overline{BQ}$ and line $\overline{BC}$	ϕ	50.7°
Surface area of Δ_{BCQ}	Area	64.1 m^2
Unit vector perpendicular to Δ_{BCQ}	$\hat{\mathbf{u}}$	0.936 $\hat{\mathbf{n}}_y$ + 0.351 $\hat{\mathbf{n}}_z$

Consider the situation when the angle between the side wall and back wall ($\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_z$) is θ (not 90°). Express the surface area of the cloth that covers that covers Δ_{BCQ} in terms θ.
Result: (Hint: Section 3.4 shows how to calculate related quantities).

$$\text{Surface area} = \frac{1}{2}\sqrt{45^2 + 120^2 \sin^2(\theta)}$$

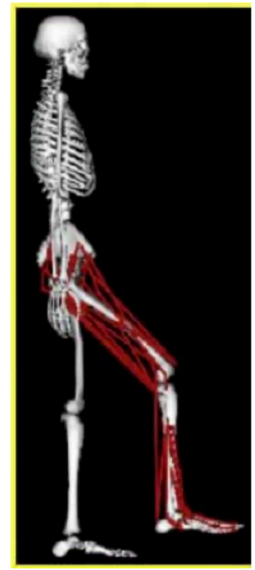
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

Calculate the surface area of the shadow of Δ_{BCQ} on the (infinite) floor when light is perpendicular to the floor. **Optional:** Recalculate when light is perpendicular to Δ_{BCQ} (assume the walls are translucent).
Result: Shadow surface area = 60 m^2 Shadow surface area = 68.4375 m^2

3.5 Ankle angle and anatomical landmarks. (Section 2.9)

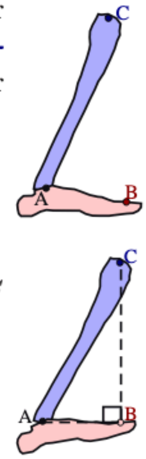
The following human lower-leg schematic has the relevant anatomical bony landmarks:

- Point A: Midpoint of the lateral and medial ***malleoli*** (near the ankle).
- Point B: Near the ***metatarsal phalangeal*** (towards the big toe).
- Point C: Midpoint of lateral and medial ***femoral condyles*** (near the knee).


Courtesy **Dr. Dan Jacobs, Dr. Gabriel Sanchez.**

Reflective markers are attached to points A, B, C to capture their positions from a point N_o (fixed in a motion-capture laboratory N). ***Motion capture*** hardware and software record the following marker data in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ fixed in N.

$$
\begin{aligned}
{}^{N_o}\vec{\mathbf{r}}^{A} &= x_A\,\widehat{\mathbf{n}}_x + y_A\,\widehat{\mathbf{n}}_y + z_A\,\widehat{\mathbf{n}}_z \\
{}^{N_o}\vec{\mathbf{r}}^{B} &= x_B\,\widehat{\mathbf{n}}_x + y_B\,\widehat{\mathbf{n}}_y + z_B\,\widehat{\mathbf{n}}_z \\
{}^{N_o}\vec{\mathbf{r}}^{C} &= x_C\,\widehat{\mathbf{n}}_x + y_C\,\widehat{\mathbf{n}}_y + z_C\,\widehat{\mathbf{n}}_z
\end{aligned}
$$



Determine the value of the "***ankle angle***" θ between ${}^{A}\vec{\mathbf{r}}^{B}$ and ${}^{A}\vec{\mathbf{r}}^{C}$ for the following special-case of a **right triangle**.

Note: This result is useful for **verifying** the more general expression.

$$
\begin{array}{lll}
x_A = 0 & y_A = 0 & z_A = 0 \\
x_B = 1 & y_B = 0 & z_B = 0 \\
x_C = 1 & y_C = 1 & z_C = 0
\end{array}
$$

Result: $\theta = 45^\circ$

Find a general expression for θ in terms of $x_A, y_A, z_A,\; x_B, y_B, z_B,\; x_C, y_C, z_C$.

Result:

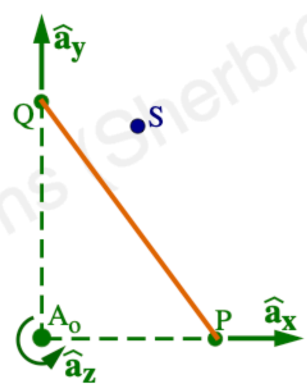
$$
\theta = \text{acos}\left\{ \frac{(x_B - x_A)(x_C - x_A) + (y_B - y_A)(y_C - y_A) + (z_B - z_A)(z_C - z_A)}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}\;\sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}} \right\}
$$

Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

3.6 †Distance from a line (2D/3D geometry). (Section 2.9).

A common geometry problem is to determine how far a point S is from a line $\overline{PQ}$. The figure to the right shows points P, Q, and S whose position vectors from a point A_o are written in terms of right-handed, orthogonal, unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$, as

$$
{}^{A_o}\vec{\mathbf{r}}^{P} = p\,\widehat{\mathbf{a}}_x \qquad {}^{A_o}\vec{\mathbf{r}}^{Q} = q\,\widehat{\mathbf{a}}_y \qquad {}^{A_o}\vec{\mathbf{r}}^{S} = s_x\,\widehat{\mathbf{a}}_x + s_y\,\widehat{\mathbf{a}}_y + s_z\,\widehat{\mathbf{a}}_z
$$



Denoting C as the point on line $\overline{PQ}$ closest to S, determine the following (in terms of p, q, s_x, s_y, s_z and symbols in the table below).

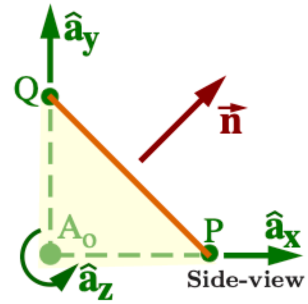
Distance between P and Q	$d^{Q/P} = \sqrt{p^2 + q^2}$
Distance between P and C	$d^{P/C} = \dfrac{q\,s_y - p\,s_x + p\,p}{d^{Q/P}}$
C's position from P	${}^{P}\vec{\mathbf{r}}^{C} = -p\,\dfrac{d^{P/C}}{d^{Q/P}}\,\widehat{\mathbf{a}}_x + q\,\dfrac{d^{P/C}}{d^{Q/P}}\,\widehat{\mathbf{a}}_y$

Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

3.7 Distance from a plane (2D geometry). (Section 2.9)

A common geometry problem is to determine how far a point is from a plane. For example, the following shows an $\hat{\mathbf{a}}_z$ side-view of a rectangular skylight window with corners at points P and Q.

The position of P and Q from A_o (a point inside the window/building) are known in terms of right-handed, orthogonal, unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$.
A street-lamp (not shown) is planned to be built near the window with a point S (the point of the street-lamp nearest the window) located as given below.



$${}^{A_o}\vec{\mathbf{r}}^{\,P} = 1\text{ m}\,\hat{\mathbf{a}}_x \qquad {}^{A_o}\vec{\mathbf{r}}^{\,Q} = 1\text{ m}\,\hat{\mathbf{a}}_y \qquad {}^{A_o}\vec{\mathbf{r}}^{\,S} = 1\text{ m}\,\hat{\mathbf{a}}_x + 0.5\text{ m}\,\hat{\mathbf{a}}_y$$

Calculate (or guess) a vector $\vec{\mathbf{n}}$ outward-normal to the skylight window.
Determine d, how far S is outside the infinite plane containing the window.

Result: ($\hat{\mathbf{n}}$ is perpendicular to the window and points outside, and d is positive if S is outside the building).

$\vec{\mathbf{n}} = 1\,\hat{\mathbf{a}}_x + 1\,\hat{\mathbf{a}}_y$ ($\vec{\mathbf{n}}$ is not a unit vector since $|\vec{\mathbf{n}}| = \sqrt{2}$) $\qquad d = \frac{0.5}{\sqrt{2}} \approx 0.35$ meters

---

Optional: Generalize these results using the symbols p, q, s_x, s_y shown below.

$${}^{A_o}\vec{\mathbf{r}}^{\,P} = p\,\hat{\mathbf{a}}_x \qquad {}^{A_o}\vec{\mathbf{r}}^{\,Q} = q\,\hat{\mathbf{a}}_y \qquad {}^{A_o}\vec{\mathbf{r}}^{\,S} = s_x\,\hat{\mathbf{a}}_x + s_y\,\hat{\mathbf{a}}_y$$

Result: Unit vector $\hat{\mathbf{n}} = \dfrac{q}{\sqrt{p^2+q^2}}\,\hat{\mathbf{a}}_x + \dfrac{p}{\sqrt{p^2+q^2}}\,\hat{\mathbf{a}}_y \qquad d = \dfrac{p\,s_y + q\,s_x - p\,q}{\sqrt{p^2+q^2}}$

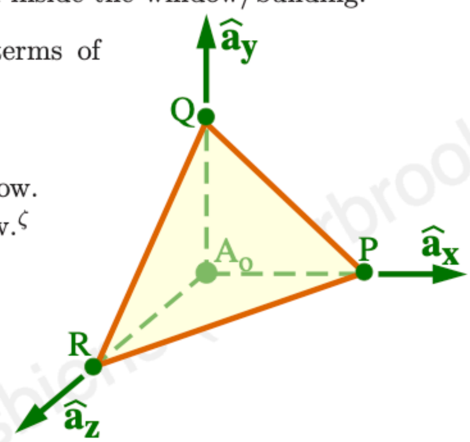
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

3.8 †Distance from a plane (3D geometry). Courtesy Dr. Adam Leeper. (Section 2.9)

A common geometry problem (in computer graphics, construction, biomechanics, engineering, etc.) is to determine the normal to a plane. For example, the following figure shows points P, Q, R located at corners of a triangular skylight window and a point A_o located inside the window/building.

The position vectors of P, Q, R from A_o are known in terms of right-handed, orthogonal, unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ as

$${}^{A_o}\vec{\mathbf{r}}^{\,P} = p\,\hat{\mathbf{a}}_x \qquad {}^{A_o}\vec{\mathbf{r}}^{\,Q} = q\,\hat{\mathbf{a}}_y \qquad {}^{A_o}\vec{\mathbf{r}}^{\,R} = r\,\hat{\mathbf{a}}_z$$



- **Draw** a unit vector $\hat{\mathbf{n}}$ outward-normal to the skylight window.
- Calculate the outward-unit-normal $\hat{\mathbf{n}}$ to the skylight window.ς

Result: (Express the unit vector $\hat{\mathbf{n}}$ in terms of p, q, r, $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$).

$$\hat{\mathbf{n}} = \frac{q\,r\,\hat{\mathbf{a}}_x + p\,r\,\hat{\mathbf{a}}_y + p\,q\,\hat{\mathbf{a}}_z}{\sqrt{p^2\,q^2 + p^2\,r^2 + q^2\,r^2}}$$

A street-lamp (not shown) is to be built outside the window.

Draw a point S (the point of the street-lamp nearest the window) at ${}^{A_o}\vec{\mathbf{r}}^{\,S} = s_x\,\hat{\mathbf{a}}_x + s_y\,\hat{\mathbf{a}}_y + s_z\,\hat{\mathbf{a}}_z$.
Determine the measure d of how far S is outside the infinite plane containing the skylight window.

Result: (A positive value of d means S is outside the building whereas a negative d means S is inside).

$$d = \frac{p\,q\,s_z + p\,r\,s_y + q\,r\,s_x - p\,q\,r}{\sqrt{p^2\,q^2 + p^2\,r^2 + q^2\,r^2}}$$

Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

Guess: When $p = q = r = 1$ m and $s_x = s_y = s_z = 0.5$ m, S is **inside/outside** the window.

Optional: Check your guess by calculating d. $\qquad d = \frac{0.5}{\sqrt{3}} \approx 0.29$ m

---

ς**Optional** (hint): List 2+ geometrical objects perpendicular to $\hat{\mathbf{n}}$ (e.g., specific planes, lines, or vectors).

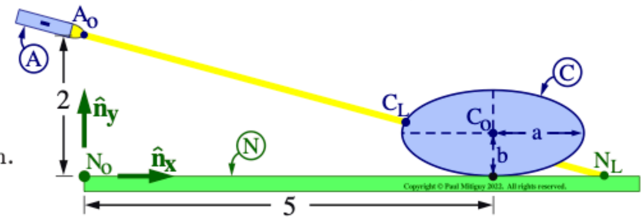
Line $\overline{RP}$	Line $\overline{RQ}$	Line $\overline{PQ}$

3.9 Laser intersection (2D geometry).

The figure shows a laser A pointing at an object with a beam that passes through a point A_o in the direction of $3\,\hat{\mathbf{n}}_x - \hat{\mathbf{n}}_y$, where unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$ are fixed in a horizontal plane N with $\hat{\mathbf{n}}_x$ horizontally-right and $\hat{\mathbf{n}}_y$ vertically-upward. A_o's position from N_o is ${}^{N_o}\vec{\mathbf{r}}^{A_o} = 2\,\hat{\mathbf{n}}_y$ (in meters).

- Determine the distance s_1 from A_o to N_L.
  N_L is the point of N that is hit by the laser beam.
- †Determine the distance s_2 from A_o to C_L.
  C_L is the point of ellipse C that is hit by the laser beam.

Results: $s_1 \approx$ 6.32 m $\qquad s_2 \approx$ 3.81 m



C is an ellipse in contact with N whose major axis is parallel to $\hat{\mathbf{n}}_x$ has semi-diameter of $a = 1.4$ m and whose minor axis has semi-diameter of $b = 0.7$ m. The position vector from N_o to C_o (C's geometric center) is ${}^{N_o}\vec{\mathbf{r}}^{C_o} = 5\,\hat{\mathbf{n}}_x + b\,\hat{\mathbf{n}}_y$.

3.10 †Laser intersection (3D geometry).

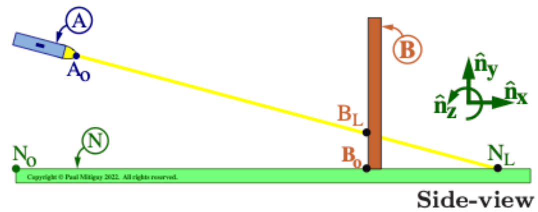
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

Each figure shows a laser A pointing at an object with a beam that passes through a point A_o in the direction of $3\,\hat{\mathbf{n}}_x - \hat{\mathbf{n}}_y + \hat{\mathbf{n}}_z$ where $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are right-handed orthogonal unit vectors fixed in a flat horizontal plane N with $\hat{\mathbf{n}}_x$ horizontally-right and $\hat{\mathbf{n}}_y$ vertically-upward. Point A_o's position vector from N_o (a point fixed in N) is ${}^{N_o}\vec{\mathbf{r}}^{A_o} = \hat{\mathbf{n}}_x + 2\,\hat{\mathbf{n}}_y$ (all lengths are in meters).

Consider a laser beam that hits point N_L of a horizontal plane N. Find the distance from A_o to N_L.

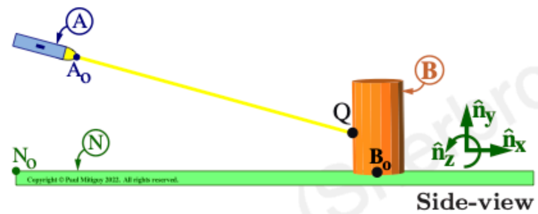
Now consider the laser beam when it hits point B_L of a vertical wall B that is in the $\hat{\mathbf{n}}_y$-$\hat{\mathbf{n}}_z$ plane (the wall is perpendicular to $\hat{\mathbf{n}}_x$). The wall is 5 m from N_o and its lower edge is parallel to $\hat{\mathbf{n}}_z$. Find the distance from A_o to B_L.

Result: $\left|{}^{A_o}\vec{\mathbf{r}}^{N_L}\right| =$ 6.63 m $\qquad \left|{}^{A_o}\vec{\mathbf{r}}^{B_L}\right| =$ 4.42 m



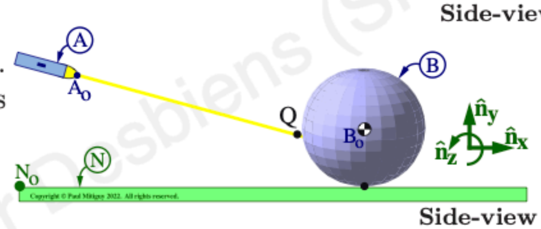
A laser beam hits point Q of a vertical cylinder B of radius 1 m. The point of B's symmetry axis in contact with N is denoted B_o and its position from N_o is ${}^{N_o}\vec{\mathbf{r}}^{B_o} = 5\,\hat{\mathbf{n}}_x + 2\,\hat{\mathbf{n}}_z$. Find the distance from A_o to Q.

Result: $\left|{}^{A_o}\vec{\mathbf{r}}^{Q}\right| =$ 3.83 m

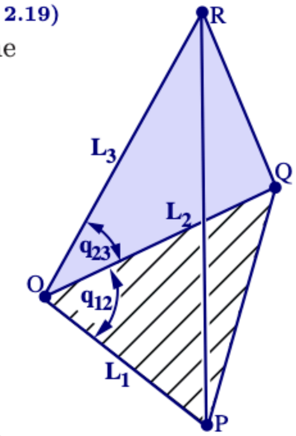


A laser beam hits point Q of a sphere B of radius 1 m. The position to the point of B in contact with N is $5\,\hat{\mathbf{n}}_x + 2\,\hat{\mathbf{n}}_z$. Find the distance from A_o to Q.

Result: $\left|{}^{A_o}\vec{\mathbf{r}}^{Q}\right| =$ 3.85 m



3.11 †Optional: Tetrahedron height, angle, volume. (Sections 2.9 and 4.6, Hw 2.19)



Consider a tetrahedron with vertices O, P, Q, R. The table below gives the lengths of three sides of the tetrahedron and the angles between those sides.

Description	Symbol	Value	Description	Symbol	Value
Length of $\overline{OP}$	L_1	1 m	Angle between $\overline{OP}$ and $\overline{OQ}$	q_{12}	60°
Length of $\overline{OQ}$	L_2	2 m	Angle between $\overline{OP}$ and $\overline{OR}$	q_{13}	55°
Length of $\overline{OR}$	L_3	3 m	Angle between $\overline{OQ}$ and $\overline{OR}$	q_{23}	50°

Tetrahedron sides are positive ($L_i > 0$) and angles $0 < q_{ij} < 180°$ ($q_{ij} = q_{12},\ q_{13},\ q_{23}$).

Form explicit expressions and calculate values for the:

- Height h of point R above the horizontal plane formed by triangle ΔOPQ
- Angle θ between plane ΔOPQ and side $\overline{OR}$ $(0 < \theta \le 90°)$.
- Calculate the area A of ΔOPQ. **Optional:** Calculate the tetrahedron's volume V.

Result: (2⁺ significant digits) Note: There are multiple methods for finding h, θ, etc.

$$h = 2.12 \text{ m} \qquad \theta = 45.12° \qquad A = \tfrac{1}{2}\, L_1\, L_2\, \sin(q_{12}) = 0.866 \text{ m}^2 \qquad V = \tfrac{1}{3}\, A\, h = 0.614 \text{ m}^3$$

$$h = \sqrt{L_3^2 - d^2} \qquad \theta = \text{acos}\left(\tfrac{d}{L_3}\right) \qquad \Big\| \qquad \theta = \text{acos}\left[\tfrac{a_{123}}{\sin(q_{12})}\right] \qquad h = L_3\, \sin(\theta)$$

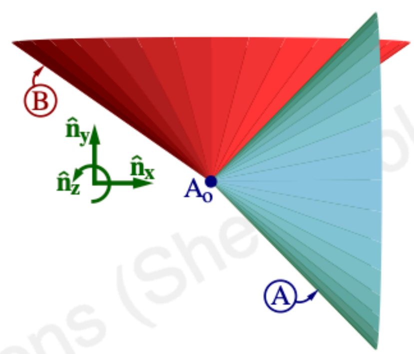
where
$$a_{123} = +\sqrt{\cos^2(q_{13}) + \cos^2(q_{23}) - 2\cos(q_{12})\cos(q_{13})\cos(q_{23})} \approx 0.611$$
$$\cos(\theta) = \frac{a_{123}}{\sin(q_{12})} \approx 0.706 \qquad d = L_3\, \frac{a_{123}}{\sin(q_{12})} = L_3 \cos(\theta) \approx 2.117 \text{ m}$$

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **2D/3D geometry.**

††**Optional:** Derive the tetrahedron volume formula V (use Calculus – not formula look-up).

††**Optional:** Derive formulas for a uniform-density tetrahedron's center of mass location and its inertia dyadic about point O. These formulas are used by CAD/CAE software for calculating mass/inertia properties of generic 3D solids. Verify the distance between point O and the tetrahedron's center of mass is 1.3 m.
Verify for a 1 kg tetrahedron, its principal moments of inertia about point O are: 0.135, 1.957, 2.023 (kg m²).

3.12 †Optional: Intersection of two cones.



Two cones A and B share a common vertex and have cone unit vectors $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$ and cone angles θ_A and θ_B, respectively, as

$$\hat{\mathbf{u}}_A = \hat{\mathbf{n}}_x \quad \theta_A = 40° \qquad\qquad \hat{\mathbf{u}}_B = \hat{\mathbf{n}}_y \quad \theta_B = 60°$$

The position vector $\vec{\mathbf{r}}_A$ from A's vertex A_o to any point on A's surface is related to $\hat{\mathbf{u}}_A$ and θ_A as shown below. Similarly for the position vector $\vec{\mathbf{r}}_B$ from B's vertex to any point on B's surface.

$$\vec{\mathbf{r}}_A \cdot \hat{\mathbf{u}}_A = |\vec{\mathbf{r}}_A| \cos(\theta_A) \qquad\qquad \vec{\mathbf{r}}_B \cdot \hat{\mathbf{u}}_B = |\vec{\mathbf{r}}_B| \cos(\theta_B)$$

In general, two cones have various ways to intersect including at a single point (shared vertex), along one line, two lines of intersection (penetration is the case shown here), and complete surface overlap.

- Find unit vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ parallel to the lines of intersection of the cones (in terms of θ_A and θ_B).
- Repeat the analysis when cone B is tilted so $\hat{\mathbf{u}}_B = \hat{\mathbf{n}}_x + \hat{\mathbf{n}}_y + \hat{\mathbf{n}}_z$ (numerical result).

Result:

	When $\hat{\mathbf{u}}_B = \hat{\mathbf{n}}_y$	When $\hat{\mathbf{u}}_B = \hat{\mathbf{n}}_x + \hat{\mathbf{n}}_y + \hat{\mathbf{n}}_z$
$\hat{\mathbf{u}}_1$	$\cos(\theta_A)\, \hat{\mathbf{n}}_x + \cos(\theta_B)\, \hat{\mathbf{n}}_y + \sqrt{1 - \cos^2(\theta_A) - \cos^2(\theta_B)}\, \hat{\mathbf{n}}_z$	$0.766\, \hat{\mathbf{n}}_x - 0.402\, \hat{\mathbf{n}}_y + 0.502\, \hat{\mathbf{n}}_z$
$\hat{\mathbf{u}}_2$	$\cos(\theta_A)\, \hat{\mathbf{n}}_x + \cos(\theta_B)\, \hat{\mathbf{n}}_y - \sqrt{1 - \cos^2(\theta_A) - \cos^2(\theta_B)}\, \hat{\mathbf{n}}_z$	$0.766\, \hat{\mathbf{n}}_x + 0.502\, \hat{\mathbf{n}}_y - 0.402\, \hat{\mathbf{n}}_z$

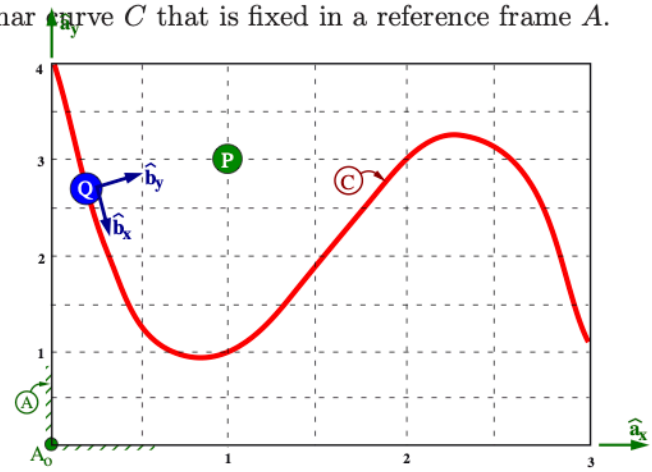
3.13 Optional: Shortest distance to a point on a planar curve

The following figure shows a particle Q on a planar curve C that is fixed in a reference frame A.

Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_{\mathrm{x}}$, $\widehat{\mathbf{a}}_{\mathrm{y}}$, $\widehat{\mathbf{a}}_{\mathrm{z}}$ are fixed in A with $\widehat{\mathbf{a}}_{\mathrm{x}}$ horizontally-right and $\widehat{\mathbf{a}}_{\mathrm{y}}$ vertically-upward. The position of Q from a point A_{o} fixed in A and the position of an arbitrary point P from A_{o} are

$$
\begin{aligned}
{}^{A_o}\vec{\mathbf{r}}^{\,Q} &= x\,\widehat{\mathbf{a}}_{\mathrm{x}} + y\,\widehat{\mathbf{a}}_{\mathrm{y}} \\
{}^{A_o}\vec{\mathbf{r}}^{\,P} &= \bar{x}\,\widehat{\mathbf{a}}_{\mathrm{x}} + \bar{y}\,\widehat{\mathbf{a}}_{\mathrm{y}}
\end{aligned}
$$

where x and y are functions of a variable s (e.g., s may stand for x, y, measure along the curve, or time).



(a) Form Q's position from P and the square of the distance d between P and Q.
Result: ${}^{P}\vec{\mathbf{r}}^{\,Q} = (x-\bar{x})\,\widehat{\mathbf{a}}_{\mathrm{x}} + (y-\bar{y})\,\widehat{\mathbf{a}}_{\mathrm{y}}$ $\qquad d^2 = (x-\bar{x})^2 + (y-\bar{y})^2$

(b) Show that the following equation governs the shortest distance from the curve to point P.
Hint: Since d is inherently positive, minimizing d^2 is identical to minimizing d.
Result:

$$(x-\bar{x})\,\frac{dx}{ds} + (y-\bar{y})\,\frac{dy}{ds} = 0$$

(c) Given a ***cubic spline*** curve of $y = 4 + \text{-}8.5\,x + 7\,x^2 + \text{-}1.5\,x^3$ and ${}^{A_o}\vec{\mathbf{r}}^{\,P} = \widehat{\mathbf{n}}_{\mathrm{x}} + 3\,\widehat{\mathbf{n}}_{\mathrm{y}}$, find the shortest distance from P to the curve and the corresponding values of x and y.
Result: distance $= 0.85875$ $\qquad x = 0.15128 \quad y = 2.869$

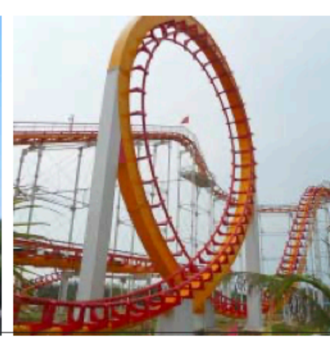
(d) Show that when Q is closest to P, ${}^{P}\vec{\mathbf{r}}^{\,Q}$ is perpendicular to the curve by showing ${}^{P}\vec{\mathbf{r}}^{\,Q} \cdot \widehat{\mathbf{b}}_{\mathrm{x}} = 0$ where $\widehat{\mathbf{b}}_{\mathrm{x}}$ is parallel to $\dfrac{{}^{A}d\,{}^{A_o}\vec{\mathbf{r}}^{\,Q}}{ds}$.

(e) **Optional:** Given the position of a particle Q on a **3D** curve from a point A_{o} that is expressed as a function of a variable s and the known position of an arbitrary point P from A_{o} as

$${}^{A_o}\vec{\mathbf{r}}^{\,Q} = x(s)\,\widehat{\mathbf{a}}_{\mathrm{x}} + y(s)\,\widehat{\mathbf{a}}_{\mathrm{y}} + z(s)\,\widehat{\mathbf{a}}_{\mathrm{z}} \qquad {}^{A_o}\vec{\mathbf{r}}^{\,P} = \bar{x}\,\widehat{\mathbf{a}}_{\mathrm{x}} + \bar{y}\,\widehat{\mathbf{a}}_{\mathrm{y}} + \bar{z}\,\widehat{\mathbf{a}}_{\mathrm{z}}$$

Show that the following equation governs the shortest distance from the curve to point P and show that when Q is closest to P, ${}^{P}\vec{\mathbf{r}}^{\,Q}$ is perpendicular to the curve by showing ${}^{P}\vec{\mathbf{r}}^{\,Q} \cdot \widehat{\mathbf{b}}_{\mathrm{x}} = 0$ where $\widehat{\mathbf{b}}_{\mathrm{x}}$ is parallel to $\dfrac{{}^{A}d\,{}^{A_o}\vec{\mathbf{r}}^{\,Q}}{ds}$.
Result:

$$(x-\bar{x})\,\frac{dx}{ds} + (y-\bar{y})\,\frac{dy}{ds} + (z-\bar{z})\,\frac{dz}{ds} = 0$$

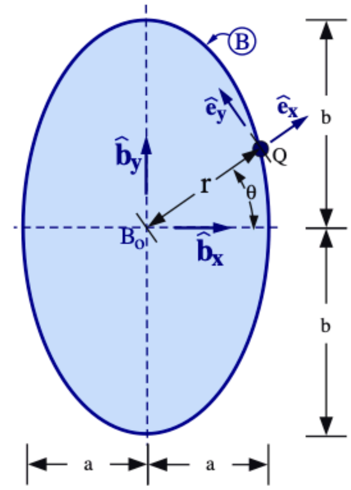


3.14 Optional: Differential geometry: Circumference, area, normal, tangent for an ellipse.

The following figure shows a point Q on the periphery of an ellipse B whose center is point B_o. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B with

- $\widehat{\mathbf{b}}_x$ horizontally-right and aligned with the ellipse's minor axis
- $\widehat{\mathbf{b}}_y$ vertically-upward and aligned with the ellipse's major axis
- $\widehat{\mathbf{b}}_z$ perpendicular to the ellipse.

Right-handed orthogonal unit vectors $\widehat{\mathbf{e}}_x$, $\widehat{\mathbf{e}}_y$, $\widehat{\mathbf{e}}_z$ are initially directed with $\widehat{\mathbf{e}}_i = \widehat{\mathbf{b}}_i$ $(i = x, y, z)$ and then are subjected to a right-handed rotation in B characterized by $\theta\,\widehat{\mathbf{b}}_z$ so $\widehat{\mathbf{e}}_x$ points from B_o to Q and $\widehat{\mathbf{e}}_z = \widehat{\mathbf{b}}_z$.



Description	Symbol	Type	Value
Half-diameter of ellipse minor axis	a	+Constant	2
Half-diameter of ellipse major axis	b	+Constant	4
Distance between N_o and Q	r	+Variable	Varies
Angle from $\widehat{\mathbf{b}}_x$ to $\widehat{\mathbf{e}}_x$ with $^+\widehat{\mathbf{b}}_z$ sense	θ	Variable	Varies

When Q's position from B_o is expressed in terms of scalars x and y as shown below-left, the equation of the ellipse can be written as shown below-right.

$$\vec{\mathbf{r}} \triangleq {}^{B_o}\vec{\mathbf{r}}^{Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(a) Referring to Sections 7.7 and 7.8, and denoting $\vec{\mathbf{r}}$ as Q's position from B_o, form $\left|{}^{B}d\vec{\mathbf{r}}\right|$ (the magnitude of the ***vector differential*** of $\vec{\mathbf{r}}$ in B). Next, write an integral (with appropriate limits) for the ellipse's circumference in terms of $d\theta$.[1] Then, express r and $\frac{dr}{d\theta}$ in terms of θ.

Result:

$$\left|{}^{B}d\vec{\mathbf{r}}\right| = \sqrt{(r\,d\theta)^2 + dr^2} \qquad \text{Circumference} = \int_{\theta=0}^{2\pi} \left|{}^{B}d\vec{\mathbf{r}}\right| = \int_{\theta=0}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta$$

$$r = \frac{a\,b}{\sqrt{a^2\,\sin^2(\theta) + b^2\,\cos^2(\theta)}} \qquad \frac{dr}{d\theta} = \frac{-a\,b\,(a^2 - b^2)\,\sin(\theta)\,\cos(\theta)}{\sqrt{a^2\,\sin^2(\theta) + b^2\,\cos^2(\theta)}^{\,3}}$$

(b) Find a formula for the circumference when $a = b$ is **constant** (the ellipse is a circle).
Calculate the circumference of an ellipse (4^+ significant digits) when $a = 2$ m and $b = 4$ m.
Express the area $d\Delta$ of the triangle formed by points B_o, $Q(\theta)$, and $Q(\theta + d\theta)$: first as a function of $\vec{\mathbf{r}}$ and $d\vec{\mathbf{r}}$; then as a function of r and θ.
Next, calculate the area of an ellipse (4^+ significant digits) when $a = 2$ m and $b = 4$ m.[2]

Result: Circumference of circle $= 2\,\pi\,b$ Circumference of ellipse $= 19.377$ m

$$d\Delta = \frac{1}{2}\left|\vec{\mathbf{r}} \times d\vec{\mathbf{r}}\right| d\theta = \frac{1}{2}\,r^2\,d\theta = \frac{1}{2}\,\frac{a^2\,b^2}{a^2\,\sin^2(\theta) + b^2\,\cos^2(\theta)}\,d\theta \qquad \text{Area of ellipse} = 25.133\ \text{m}^2$$

(c) There is an exact closed-form solution for the circumference of an ellipse. **True/[False]**.
There is an exact closed-form solution for the area of an ellipse. **[True]/False**.

(d) **Draw** an outward normal vector $\vec{\mathbf{n}}$ and tangent vector $\vec{\mathbf{t}}$ at point Q.
Express $\vec{\mathbf{n}}$ and $\vec{\mathbf{t}}$ in terms of $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ when $x = \frac{a}{2}$.

Result:

Ellipse: $a = 2$ and $b = 4$	Circle: $a = 2$ and $b = 2$
$\vec{\mathbf{n}} = 1.0\,\widehat{\mathbf{b}}_x + 0.433\,\widehat{\mathbf{b}}_y$	$\vec{\mathbf{n}} = 0.5\,\widehat{\mathbf{b}}_x + 0.866\,\widehat{\mathbf{b}}_y$
$\vec{\mathbf{t}} = -0.433\,\widehat{\mathbf{b}}_x + 1.0\,\widehat{\mathbf{b}}_y$	$\vec{\mathbf{t}} = -0.866\,\widehat{\mathbf{b}}_x + 0.5\,\widehat{\mathbf{b}}_y$

When $a = b$ (the ellipse is a circle), $\vec{\mathbf{n}}$ is always parallel to ${}^{Q/B_o}\vec{\mathbf{r}}$ **[True]/False**
When $a \neq b$ (the ellipse is not a circle), $\vec{\mathbf{n}}$ is always parallel to ${}^{Q/B_o}\vec{\mathbf{r}}$ **True/[False]**

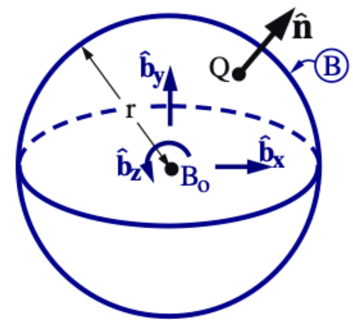
[1] The integral simplifies by noting that $d\theta$ is **positive** when the integral's upper-limit is larger than the integral's lower-limit.
[2] Verify your numerical integration result with the following formula for the area of ellipse: Area $= \pi\,a\,b$.

(e) **Optional:** Show how the definition of an ***ellipse*** results in $F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

3.15 Optional: Normal to a sphere

A ***sphere*** can be defined as the locus of points that are a distance r (called the ***sphere's radius***) from a point B_o (called the ***sphere's center***).

The figure to the right shows a sphere of radius r that is centered at point B_o. The position of a point Q on the sphere's periphery from centroid B_o can be expressed in terms of scalars x and y and right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ as



$$^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y + z\,\widehat{\mathbf{b}}_z$$

- Show how a sphere's definition results in the following relationship.

Result: $F(x,y,z) = x^2 + y^2 + z^2 - r^2 = 0$

When a scalar function F describes an object's boundary, the spatial gradient $\vec{\nabla} F$ is normal to the boundary. With $^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y + z\,\widehat{\mathbf{b}}_y$, $\vec{\nabla} F$ can be expressed as

$$\vec{\nabla} F \underset{(7.13)}{=} \frac{\partial F}{\partial x}\,\widehat{\mathbf{b}}_x + \frac{\partial F}{\partial y}\,\widehat{\mathbf{b}}_y + \frac{\partial F}{\partial z}\,\widehat{\mathbf{b}}_z$$

- Use $\vec{\nabla} F$ to calculate an outward normal vector $\vec{\mathbf{n}}$ at point Q in terms of x, y, r, etc.

Result: $\vec{\mathbf{n}} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y + z\,\widehat{\mathbf{b}}_y$

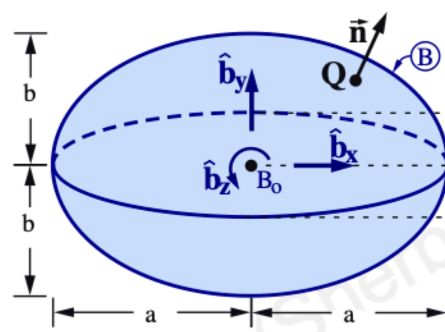
3.16 Optional: Normal to an ellipsoid

The following figure shows a point Q on a ellipsoid of semi-diameters a, b, c.

Right-handed, orthogonal, unit vectors $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$ are directed with $\widehat{\mathbf{b}}_x$ pointing right along the ellipsoid's major axis and $\widehat{\mathbf{b}}_y$ pointing up along the ellipsoid's minor axis.

The position of Q from B_o (the ellipsoid's center) can be expressed in terms of the scalars x, y, z as

$$^{B_o}\vec{\mathbf{r}}^{\,Q} = x\,\widehat{\mathbf{b}}_x + y\,\widehat{\mathbf{b}}_y + z\,\widehat{\mathbf{b}}_z$$



- Calculate x, y, z when $x = \frac{a}{2}$, $z = \frac{c}{2}$, $a = 3$, $b = c = 2$.

Result: $x = 1.5$ $\quad y = 2\sqrt{\frac{1}{2}} \approx 1.414214$ $\quad z = 1$

- Determine an outward normal vector $\vec{\mathbf{n}}$ at point Q in terms of x, y, z, a, b, c.
- Calculate the unit vector in the $\vec{\mathbf{n}}$ direction when $x = \frac{a}{2}$, $z = \frac{c}{2}$, $a = 3$, $c = 2$.

Result:

General case	Unit vector with numerical values
$\vec{\mathbf{n}} = \frac{2\,x}{a^2}\,\widehat{\mathbf{b}}_x + \frac{2\,y}{b^2}\,\widehat{\mathbf{b}}_y + \frac{2\,z}{c^2}\,\widehat{\mathbf{b}}_z$	$\widehat{\mathbf{n}} = 0.359\,\widehat{\mathbf{b}}_x + 0.762\,\widehat{\mathbf{b}}_y + 0.539\,\widehat{\mathbf{b}}_z$