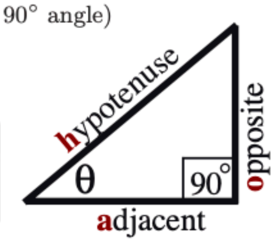
4.1 ♣ (1900 BC). Sine, cosine, tangent as ratios of sides of a right triangle. (Section 1.4)

Below is a ***right triangle*** (triangle with a 90° angle) with one angle labeled as θ. Write definitions for sine, cosine, and tangent in terms of:

- **h**ypotenuse – the triangle's longest side (opposite the 90° angle)
- **o**pposite – the side opposite to θ
- **a**djacent – the side adjacent to θ

I can draw a triangle with a negative-length side	True/[False]
Using the **limited** definition shown right, $\sin(\theta)$ (the sine of an angle) can be negative.	True/[False]

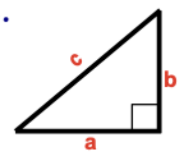


Memorize: ***Soh Cah Toa***

$$\sin(\theta) \triangleq \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) \triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) \triangleq \frac{\text{opposite}}{\text{adjacent}}$$

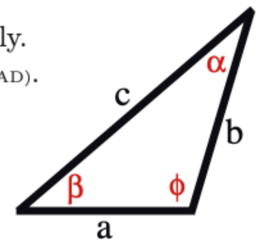
4.2 ♣ (1900 BC - 1400 AD) Pythagorean theorem and law of cosines. (Section 1.4.2).

Draw a right-triangle with a hypotenuse of length c and other sides of length a and b. Relate c to a and b with the ***Pythagorean theorem***.



Result: Babylonians 1900 BC to Pythagoreus 525 BC. $c^2 = a^2 + b^2$ ***memorize***

Shown right is a triangle with angles α, β, ϕ opposite sides a, b, c, respectively. Complete each formula below using the ***law of cosines*** (Euclid 300 BC - Al-Kashi 1400 AD).



Result: $c^2 = a^2 + b^2 - 2\,a\,b\cos(\phi)$ ***memorize***

$$a^2 = b^2 + c^2 - 2\,b\,c\cos(\alpha)$$
$$b^2 = c^2 + a^2 - 2\,c\,a\cos(\beta)$$

The ***Pythagorean theorem*** is a special case of the ***law of cosines***. [True]/False. (circle one).

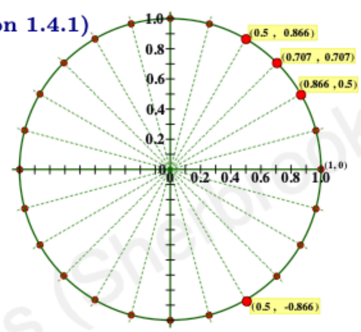
4.3 ♣ (140 BC - 1500 AD) Unit circle concept of sine and cosine. (Section 1.4.1)

Angle θ	$\sin(\theta)$	$\cos(\theta)$
0°	0	1
30°	0.5	≈ 0.866
45°	≈ 0.707	≈ 0.707
60°	≈ 0.866	0.5
90°	1	0
120°	≈ 0.866	0.5
150°	0.5	≈ -0.866

Label the blanked coordinates on the unit circle to the right.

Note: The unit circle expands the concepts of sine and cosine to **negative** values and its tabulated values provide data for Euler's graphs.

Note: Negative numbers were invented ≈ 650 AD, developed 900 AD – 1200 AD, and widely adopted 1500 AD.

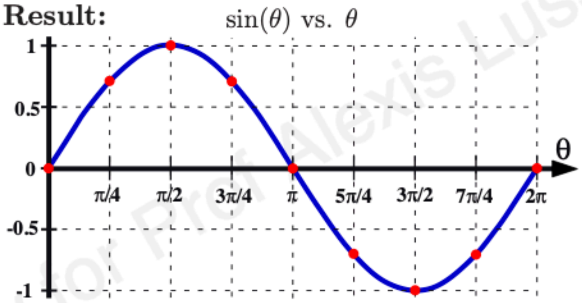


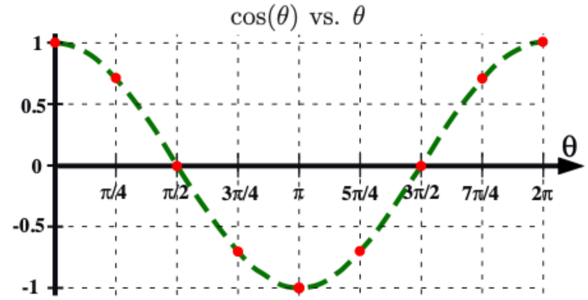
The triangle definition of sine and cosine in Hw 4.1 results in	$0° < \theta < 90°$	$0 < \sin(\theta) < 1$	$0 < \cos(\theta) < 1$
The unit circle extends the range for θ and sine and cosine to	$0° \le \theta \le 360°$	$-1 \le \sin(\theta) \le 1$	$-1 \le \cos(\theta) \le 1$

4.4 ♣ (Euler 1730 AD) Sine and cosine as functions. (Section 1.4.3)

Graph sine and cosine as functions of the angle θ over the range $0 \le \theta \le 2\,\pi$ radians.
Note: In 1730 A.D., Euler invented the sine and cosine **functions** (more than just ratios of sides of a triangle).

Result:





4.5 ♣ Ranges for arguments and return values for inverse trigonometric functions.

Determine all real return values and argument values for the following **real** trigonometric and inverse-trigonometric functions in computer languages such as Java, C^{++}, MATLAB®, MotionGenesis, ...

Range of return values for z	Function	Range of argument values for x	Note
$-1 \le z \le 1$	$z = \cos(x)$	$-\infty < x < \infty$	
$-1 \le z \le 1$	$z = \sin(x)$	$-\infty < x < \infty$	
$-\infty < z < \infty$	$z = \tan(x)$	$-\infty < x < \infty$	$x \ne \frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \ldots$
$0 \le z \le \pi$	$z = \text{acos}(x)$	$-1 \le x \le 1$	
$-\pi/2 \le z \le \pi/2$	$z = \text{asin}(x)$	$-1 \le x \le 1$	
$-\pi/2 < z < \pi/2$	$z = \text{atan}(x)$	$-\infty < x < \infty$	
$-\pi \le z \le \pi$	$z = \texttt{atan2}(y, x)$	$-\infty < y < \infty$ $-\infty < x < \infty$	$\texttt{atan2}(0,0)$ is undefined

4.6 ♣ What is an angle? (Section 5.7).

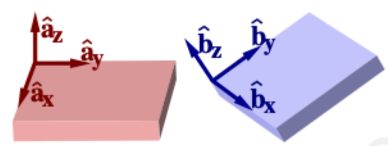
Draw the "geometry equipment" listed in the 1^{st} column of the following table. Complete the 2^{nd} column with appropriate ranges for the angle θ (in degrees).

"Geometry equipment"	Draw	Appropriate range for θ
2 lines		$0° \le \theta \le 90°$
Vector and line		$0° \le \theta \le 90°$
2 vectors		$0° \le \theta \le 180°$
2 vectors and a sense of positive rotation		$-180° < \theta \le 180°$
2 vectors, a sense of $\pm$ rotation, and time-history/continuity	Not applicable	$-\infty° < \theta < \infty°$

4.7 ${}^aR^b$ for dot-products, cross-products, and angles between vectors. (Section 5.4).

The ${}^aR^b$ rotation table relates two sets of right-handed, orthogonal, unit vectors, namely $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

${}^aR^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{a}_x$	0.962	-0.084	0.259
$\hat{a}_y$	0.170	0.928	-0.330
$\hat{a}_z$	-0.212	0.362	0.908



Efficiently determine the following dot-products and angles between vectors (2^+ significant digits). Then perform the calculations involving $\vec{v}_1 = 2\,\hat{a}_x$ and $\vec{v}_2 = \hat{a}_x + \hat{b}_x$. **Show work.**

$\hat{a}_x \cdot \hat{a}_x = 1$	$\angle(\hat{a}_x, \hat{a}_x) = 0°$	$\hat{a}_y \cdot \hat{a}_z = 0$	$\angle(\hat{a}_y, \hat{a}_x) = 90°$
$\hat{b}_z \cdot \hat{b}_y = 0$	$\angle(\hat{b}_z, \hat{b}_y) = 90°$	$\hat{a}_x \cdot \hat{b}_x = 0.962$	$\angle(\hat{a}_x, \hat{b}_x) = 15.85°$
$\hat{a}_x \cdot \hat{b}_y = -0.084$	$\angle(\hat{a}_x, \hat{b}_y) = 94.8°$	$\vec{v}_1 \cdot \vec{v}_2 \approx 3.9$	$\angle(\vec{v}_1, \vec{v}_2) \approx 7.9°$

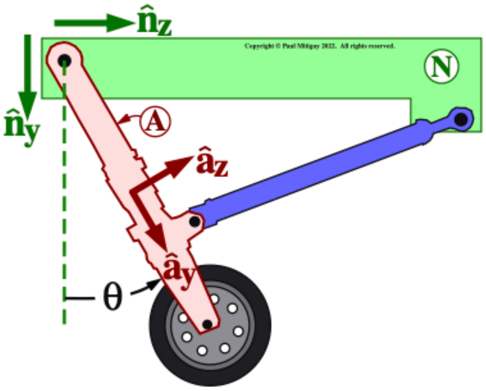
Result: $\vec{v}_1 \times \vec{v}_2 = 0.5176\,\hat{b}_y + 0.1684\,\hat{b}_z = 0.425\,\hat{a}_y + 0.3402\,\hat{a}_z$

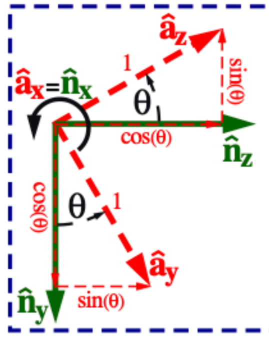
Express the unit vector $\hat{u}$ in the direction of $3\,\hat{a}_z + 4\,\hat{b}_z$ in terms of $\hat{a}_z$ and $\hat{b}_z$.
Express $\vec{v} = \hat{a}_y + \hat{b}_y$ in terms of $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.

Result: $\hat{u} = 0.4386\,\hat{a}_z + 0.5848\,\hat{b}_z$ $\quad \vec{v} = -0.0842\,\hat{a}_x + 1.928\,\hat{a}_y + 0.3619\,\hat{a}_z$

4.8 ♣ SohCahToa: Rotation tables for a landing gear system. (Section 5.5).

The figures below show three versions of the same landing gear system with a strut A that has a simple rotation relative to a fuselage N. Each figure has a different orientation for right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ (fixed in N) and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ (fixed in A). **Redraw** $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ so it is **easy to see a right-triangle** with sines and cosines. Express each of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$, then form the ${}^aR^n$ rotation table for each figure.[1] Next, form ${}^N\vec{\boldsymbol{\omega}}^A$ (A's angular velocity in N) in terms of $\dot\theta$ and one of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$.

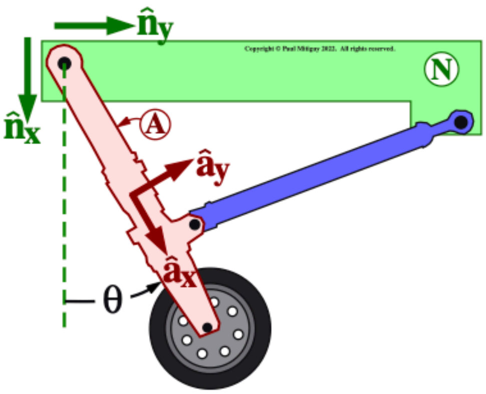


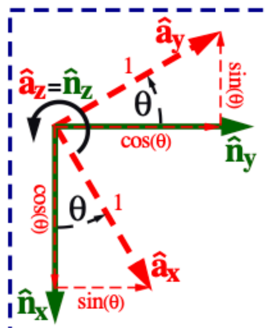


$$\hat{\mathbf{a}}_x = \hat{\mathbf{n}}_x$$
$$\hat{\mathbf{a}}_y = \cos(\theta)\,\hat{\mathbf{n}}_y + \sin(\theta)\,\hat{\mathbf{n}}_z$$
$$\hat{\mathbf{a}}_z = \text{-}\sin(\theta)\,\hat{\mathbf{n}}_y + \cos(\theta)\,\hat{\mathbf{n}}_z$$

${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	1	0	0
$\hat{\mathbf{a}}_y$	0	$\cos(\theta)$	$\sin(\theta)$
$\hat{\mathbf{a}}_z$	0	$\text{-}\sin(\theta)$	$\cos(\theta)$

$${}^N\vec{\boldsymbol{\omega}}^A = \dot\theta\,\hat{\mathbf{a}}_x$$

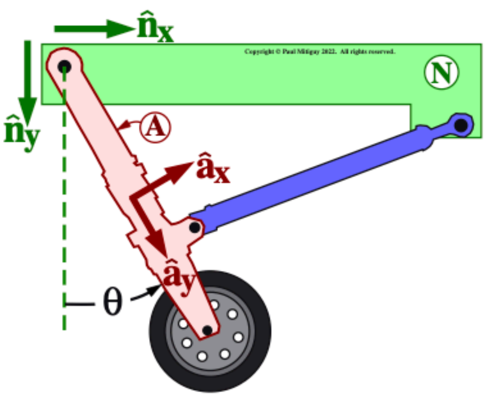


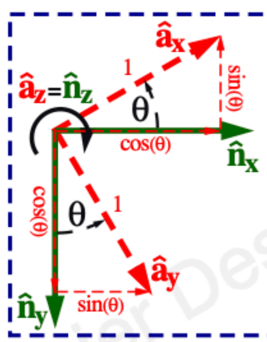


$$\hat{\mathbf{a}}_x = \cos(\theta)\,\hat{\mathbf{n}}_x + \sin(\theta)\,\hat{\mathbf{n}}_y$$
$$\hat{\mathbf{a}}_y = \text{-}\sin(\theta)\,\hat{\mathbf{n}}_x + \cos(\theta)\,\hat{\mathbf{n}}_y$$
$$\hat{\mathbf{a}}_z = \hat{\mathbf{n}}_z$$

${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{\mathbf{a}}_y$	$\text{-}\sin(\theta)$	$\cos(\theta)$	0
$\hat{\mathbf{a}}_z$	0	0	1

$${}^N\vec{\boldsymbol{\omega}}^A = \dot\theta\,\hat{\mathbf{a}}_z$$

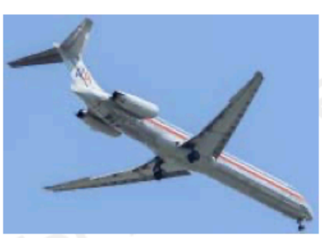


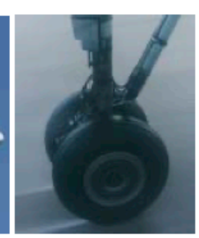


Show work to express each of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.

${}^aR^n$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	$\cos(\theta)$	$\text{-}\sin(\theta)$	0
$\hat{\mathbf{a}}_y$	$\sin(\theta)$	$\cos(\theta)$	0
$\hat{\mathbf{a}}_z$	0	0	1

$${}^N\vec{\boldsymbol{\omega}}^A = \text{-}\dot\theta\,\hat{\mathbf{a}}_z$$





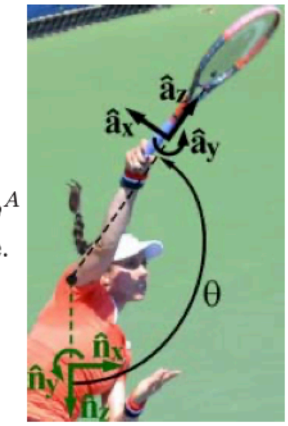
[1] Each figure has two missing vectors (e.g., $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{a}}_x$ are missing from the first figure). Use the fact that each set of vectors is **right-handed** to add the missing vectors to each figure.

4.9 ♣ Rotation table for a tennis swing. (Section 5.5). .

Shown right are two sets of right-handed orthogonal unit vectors, namely:
$\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ (fixed on the tennis court) and $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ (fixed on the athlete's arm).

$\widehat{\mathbf{n}}_x$ is horizontal (from baseline to net) $\widehat{\mathbf{n}}_z$ is vertically downward

$\widehat{\mathbf{a}}_x$ is along the arm (from shoulder to hand) $\widehat{\mathbf{a}}_y = \widehat{\mathbf{n}}_y$



Form the rotation table relating basis A ($\widehat{\mathbf{a}}_x, \widehat{\mathbf{a}}_y, \widehat{\mathbf{a}}_z$) to basis N ($\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$) and ${}^N\vec{\boldsymbol{\omega}}^A$ (A's angular velocity in N) in terms of the angle θ from $\widehat{\mathbf{n}}_z$ to $\widehat{\mathbf{a}}_z$ with $+\widehat{\mathbf{n}}_y$ sense.

Result:

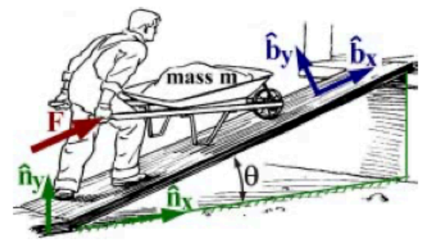
Redraw $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
Write $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.
Use sine, cosine, and vector addition (i.e., resolve vectors into components).

${}^aR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{a}}_x$	$\cos(\theta)$	0	$-\sin(\theta)$
$\widehat{\mathbf{a}}_y$	0	1	0
$\widehat{\mathbf{a}}_z$	$\sin(\theta)$	0	$\cos(\theta)$

$${}^N\vec{\boldsymbol{\omega}}^A = \dot{\theta}\,\widehat{\mathbf{a}}_y$$

4.10 FE/EIT – Block on frictionless inclined plane (2D analysis).

A man pushes a wheelbarrow up a plank inclined by angle θ. Unit vector $\widehat{\mathbf{n}}_x$ is horizontally-right, $\widehat{\mathbf{n}}_y$ is vertically-upward, $\widehat{\mathbf{b}}_x$ is along the plank, and $\widehat{\mathbf{b}}_y$ is normal to the plank. The wheelbarrow is modeled as a particle P of mass m in **frictionless** contact with the plank. The man pushes the wheelbarrow with a force $F\,\widehat{\mathbf{b}}_x$.



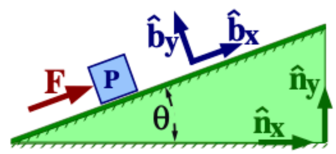
- Relate $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$ to $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$ and complete the associated rotation table.

Result:

Redraw $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$ in a useful way.

$$\widehat{\mathbf{b}}_x = \cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y$$
$$\widehat{\mathbf{b}}_y = -\sin(\theta)\,\widehat{\mathbf{n}}_x + \cos(\theta)\,\widehat{\mathbf{n}}_y \quad \Rightarrow$$

${}^bR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$

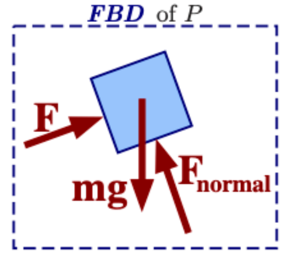


- **Draw** P's ***free-body diagram (FBD)*** and form the net force on P.

Result: in terms of F, m, g (Earth's gravitational acceleration) and your own symbols.

$$\vec{\mathbf{F}}_{\text{Net}} = F\,\widehat{\mathbf{b}}_x + F_{\text{normal}}\,\widehat{\mathbf{b}}_y - m\,g\,\widehat{\mathbf{n}}_y$$

FBD of *P*



- Knowing P is stationary, set $\vec{\mathbf{F}}_{\text{Net}} = \vec{\mathbf{0}}$ (*static equilibrium*) and solve for F.

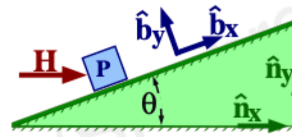
Hint: Form $\vec{\mathbf{F}}_{\text{Net}} \cdot \widehat{\mathbf{u}} = 0$, where $\widehat{\mathbf{u}}$ is a cleverly chosen unit vector.

- Repeat the analysis when the man instead applies a horizontal force $H\,\widehat{\mathbf{n}}_x$.

Hint: **Cleverly** choose $\widehat{\mathbf{u}}$ so the normal force does **not** appear in the scalar equation.

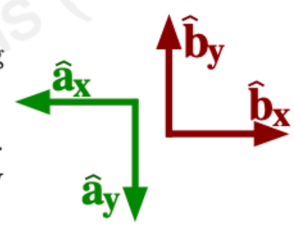
Result: (only in terms of m, g, θ).

(with only $F\,\widehat{\mathbf{b}}_x$) $F = m\,g\,\sin(\theta)$ (with only $H\,\widehat{\mathbf{n}}_x$) $H = m\,g\,\tan(\theta)$



4.11 3D visual thinking (draw/think 3D) - for disorderly unit vectors

Shown right are parts of two right-handed orthogonal bases A and B consisting of unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, respectively.



The bases are initially oriented with $\theta = 0$, $\widehat{\mathbf{b}}_x = -\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{b}}_y = -\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{b}}_z = \widehat{\mathbf{a}}_z$. Basis B is then subjected to a right-handed rotation relative to A in each way described below. Express each ${}^bR^a$ rotation matrix below in terms of θ.

Rotation of B in A characterized by $+\theta\,\widehat{\mathbf{a}}_z$ (θ is the angle from $-\widehat{\mathbf{a}}_x$ to $\widehat{\mathbf{b}}_x$ with $+\widehat{\mathbf{a}}_z$ sense).

${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$	$-\cos(\theta)$	$-\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$\sin(\theta)$	$-\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

Rotation of B in A characterized by $+\theta\,\widehat{\mathbf{a}}_y$ (θ is the angle from $\widehat{\mathbf{a}}_z$ to $\widehat{\mathbf{b}}_z$ with $+\widehat{\mathbf{a}}_y$ sense).

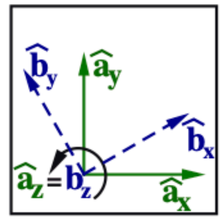
${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$	$-\cos(\theta)$	0	$\sin(\theta)$
$\widehat{\mathbf{b}}_y$	0	-1	0
$\widehat{\mathbf{b}}_z$	$\sin(\theta)$	0	$\cos(\theta)$

4.12 ♣ Rotation table concepts: What is an angle. (Section 5.7)

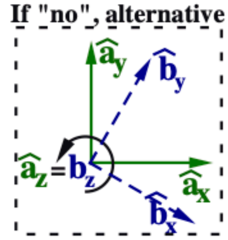
Given: Two sets of right-handed orthogonal unitary bases $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

- Determine each element of the ${}^bR^a$ rotation table below so $\hat{b}_z = \hat{a}_z$ and the angle between $\hat{b}_x$ and $\hat{a}_x$ is 30°. **Draw** $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ (clearly show relative orientation of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ to $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$).
- Are there other orientations of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ such that $\hat{b}_z = \hat{a}_z$ and the angle between $\hat{b}_x$ and $\hat{a}_x$ is 30°? **Yes**/**No**.
- Is ${}^bR^a$ unique when $\hat{b}_z = \hat{a}_z$ and $\hat{b}_x \cdot \hat{a}_x = \frac{\sqrt{3}}{2}$? **Yes**/**No**.
  If **No**, **draw** an alternative orientation for $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$.

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	cos(30°)	sin(30°)	0
$\hat{b}_y$	-sin(30°)	cos(30°)	0
$\hat{b}_z$	0	0	1



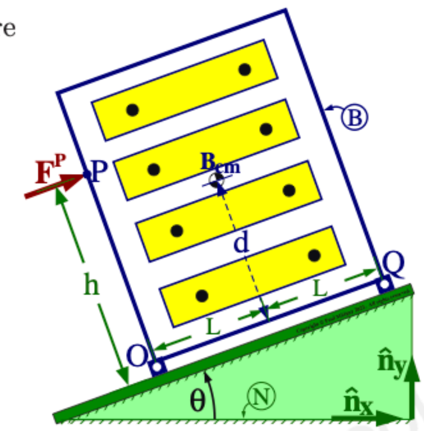
If "no", alternative



4.13 FE/EIT – Bureau sliding on a smooth inclined plane (2D analysis, frictionless).

A rigid uniform-density bureau B is in **frictionless** contact with an inclined plane at points O and Q of B. A force of magnitude F^P is applied at point P of B (the force is directed up the inclined plane).

Right-handed orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ and $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are directed with $\hat{n}_x$ horizontally-right, $\hat{n}_y$ vertically-upward, $\hat{b}_x$ from O to Q, $\hat{b}_y$ from O to P, and $\hat{b}_z = \hat{n}_z$.



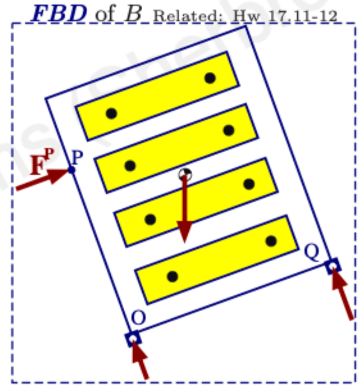
Angle from $\hat{n}_x$ to $\hat{b}_x$ with $+\hat{n}_z$ sense	θ
Mass of bureau	m
Earth's gravitational acceleration	g
Half-width of bureau	L
Distance between points O and P	h
Distance between B_{cm} and line $\overline{OQ}$	d
$\hat{b}_x$ measure of force on B from person	F^P
$\hat{b}_y$ measure of normal force on O from inclined plane	F_y^O
$\hat{b}_y$ measure of normal force on Q from inclined plane	F_y^Q

- **Draw** $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and form the ${}^bR^n$ rotation table.
- **Draw** the bureau's ***free-body diagram (FBD)*** and form the net force on B and moment of all forces on B about O.

Result:

${}^bR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{b}_x$	cos(θ)	sin(θ)	0
$\hat{b}_y$	-sin(θ)	cos(θ)	0
$\hat{b}_z$	0	0	1

FBD of B Related: Hw 17.11-12



$$\vec{F}_{Net} = F^P\,\hat{b}_x + (F_y^O + F_y^Q)\,\hat{b}_y - m\,g\,\hat{n}_y$$

$$\vec{M}^{B/O} = \left\{ m\,g\,[\,d\,\sin(\theta) - L\,\cos(\theta)] + 2\,L\,F_y^Q - h\,F^P \right\}\hat{b}_z$$

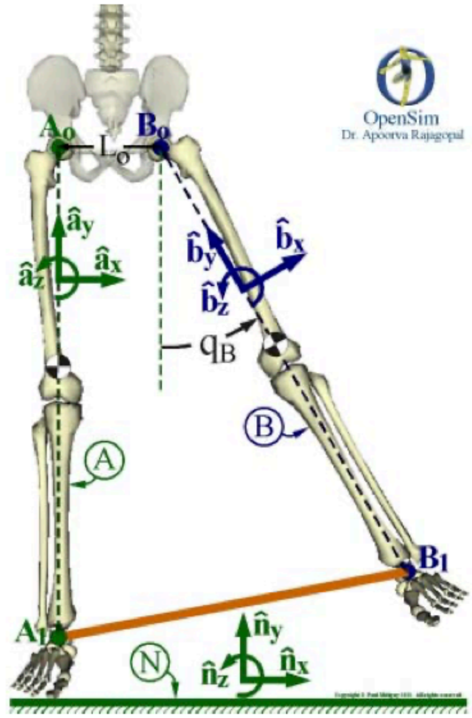
- Knowing B slides at **constant** speed, solve for F^P, F_y^O, F_y^Q by setting $\vec{F}_{Net} = \vec{0}$ and $\vec{M}^{B/O} = \vec{0}$.

Result: (*static equilibrium*). Solution at www.MotionGenesis.com ⇒ Get Started ⇒ Simple statics.

$$F^P = m\,g\,\sin(\theta) \qquad F_y^O = \frac{m\,g}{2}\left[\cos(\theta) - \frac{h-d}{L}\sin(\theta)\right] \qquad F_y^Q = \frac{m\,g}{2}\left[\cos(\theta) + \frac{h-d}{L}\sin(\theta)\right]$$

4.14 Hip abduction geometry (2D analysis). (Section 5.5)

The schematic shows the lower-half of a person doing ***hip abduction***[a] exercises which stretch an exercise-band that connects point A_1 of leg A to point B_1 of leg B. Leg A and the pelvis/hips are stationary relative to ground N, whereas leg B rotates in N by a ***hip abduction angle*** q_B.



Orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in legs A and B, respectively, with:

$\hat{\mathbf{a}}_x$ directed from A_o to B_o (horizontally-right)
$\hat{\mathbf{a}}_y$ directed from A_1 to A_o (vertically-upward)
$\hat{\mathbf{b}}_y$ directed from B_1 to B_o
$\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z$ perpendicular to the plane in which B moves.

- Form the ${}^bR^a$ rotation table and B's angular velocity in N.

Result: (in terms of q_B, the angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{b}}_y$ with $+\hat{\mathbf{a}}_z$ sense).

${}^bR^a$	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{b}}_x$	$\cos(q_B)$	$\sin(q_B)$	0
$\hat{\mathbf{b}}_y$	$-\sin(q_B)$	$\cos(q_B)$	0
$\hat{\mathbf{b}}_z$	0	0	1

$${}^N\vec{\boldsymbol{\omega}}^B = \dot{q}_B\,\hat{\mathbf{b}}_z$$

[a] ***Abduction*** is a general medical term for movement of a limb away from the centerline of the body.

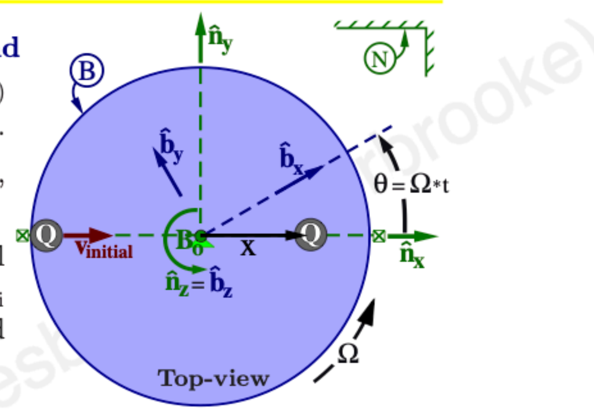
- Denoting L_{leg} as the distance between A_o and A_1 (and B_o and B_1) and L_o as the distance between A_o and B_o, form ${}^{A_1}\vec{\mathbf{r}}^{B_2}$ (B_1's position from A_1) in terms of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{b}}_y$. Next, form the stretched-length d of the exercise-band. Hint: From Section 3.6, the distance between A_1 and B_1 is $d = \sqrt{{}^{A_1}\vec{\mathbf{r}}^{B_1} \cdot {}^{A_1}\vec{\mathbf{r}}^{B_1}}$.

Result: (in terms of L_{leg}, L_o, q_B). Reminder: the ${}^bR^a$ rotation table makes it easy to do dot-products between $\hat{\mathbf{a}}_i$ and $\hat{\mathbf{b}}_j$.

$${}^{A_1}\vec{\mathbf{r}}^{B_1} = L_{\text{leg}}\,\hat{\mathbf{a}}_y + L_o\,\hat{\mathbf{a}}_x - L_{\text{leg}}\,\hat{\mathbf{b}}_y \qquad d = \sqrt{2\,L_{\text{leg}}^2\,[1 - \cos(q_B)] + 2\,L_{\text{leg}}\,L_o\,\sin(q_B) + L_o^2}$$

4.15 Motion trajectory of a hockey-puck on a merry-go round

The figure shows the top-view of a merry-go-round (circular disk) B that spins with a constant angular speed Ω on Earth (frame N).



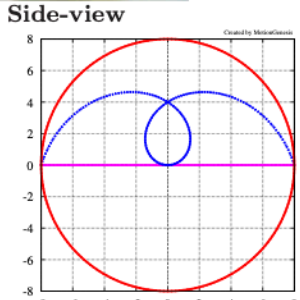
Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are fixed in N, with $\hat{\mathbf{n}}_x$ East, $\hat{\mathbf{n}}_y$ North, and $\hat{\mathbf{n}}_z$ vertically-upward.

B_o (B's geometric center) is fixed on N. Right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in B. At time $t = 0$, $\hat{\mathbf{b}}_i = \hat{\mathbf{n}}_i$ ($i = x, y, z$). B's orientation is characterized by a right-handed rotation in N characterized by $\theta\,\hat{\mathbf{n}}_z$ (hence, $\theta = \Omega\,t$).

A hockey puck (particle) Q is slid across the snow-covered merry-go-round. As viewed from frame N, Q travels directly East in a straight line so that Q's position from B_o is ${}^{B_o}\vec{\mathbf{r}}^{Q} = x\,\hat{\mathbf{n}}_x$.

Side-view

- To trace Q's trajectory (path) in the snow on top of merry-go-round B, it helps to express ${}^{B_o}\vec{\mathbf{r}}^{Q} = x_B\,\hat{\mathbf{b}}_x + y_B\,\hat{\mathbf{b}}_y$. Form x_B, y_B in terms of x, θ. Using $x = -8 + 4\,t$ and $\Omega = \frac{\pi}{2}\,\frac{\text{rad}}{\text{sec}}$, plot Q's path on B in one turn (4 seconds).



Result:

$$x_B = x\,\cos(\theta) \qquad y_B = -x\,\sin(\theta)$$

- †**Optional:** Determine how far Q travels on N and on B in one turn.

Result: Distance on N = 16.00 m   Distance on B ≈ 31.12 m

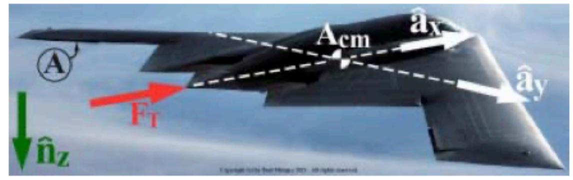
4.16 FE/EIT – Aircraft force equilibrium (2D analysis, coincident mass/aerodynamic centers).

The following figure shows an aircraft A in flight over Earth (a Newtonian reference frame N). Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in A and N, respectively, with

$\widehat{\mathbf{a}}_x$ from the aircraft's tail towards its nose

$\widehat{\mathbf{a}}_y = \widehat{\mathbf{n}}_y$ outward along the right-wing
(perpendicular to the plane in which A_{cm} moves)

$\widehat{\mathbf{n}}_z$ vertically-downward



A's orientation in N is determined by initially setting $\widehat{\mathbf{a}}_i = \widehat{\mathbf{n}}_i$ $(i = x, y, z)$ and then subjecting A to a right-handed rotation in N characterized by $\theta\,\widehat{\mathbf{a}}_y$ (θ is the aircraft's "pitch angle", regarded as positive when the nose is up).

- Form the ${}^aR^n$ rotation table in terms of θ.
- Form ${}^N\vec{\boldsymbol{\omega}}^A$, A's angular velocity in N.

${}^aR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{a}}_x$	$\cos(\theta)$	0	$-\sin(\theta)$
$\widehat{\mathbf{a}}_y$	0	0	1
$\widehat{\mathbf{a}}_z$	$\sin(\theta)$	0	$\cos(\theta)$

$${}^N\vec{\boldsymbol{\omega}}^A = \dot{\theta}\,\widehat{\mathbf{a}}_y$$

Continuously distributed aerodynamic forces on A are replaced with a couple of torque $\vec{\mathbf{T}}^A$ together with a lift and drag force $\vec{\mathbf{F}}_{Lift}$ and $\vec{\mathbf{F}}_{Drag}$ applied at A's center of mass A_{cm}.[2] Forces from the aircraft's jet engines are replaced by a force $\vec{\mathbf{F}}_T\,\widehat{\mathbf{a}}_x$ at the aircraft's tail. Earth's gravitational forces on A are replaced with a force $m\,g\,\widehat{\mathbf{n}}_z$ applied to A's center of mass.

$$\vec{\mathbf{F}}_{Lift} = -F_L\,\widehat{\mathbf{a}}_z \qquad \vec{\mathbf{F}}_{Drag} = -F_D\,\widehat{\mathbf{a}}_x \qquad \vec{\mathbf{T}}^A \approx \vec{\mathbf{0}}$$
$$\vec{\mathbf{F}}_{thrust} = F_T\,\widehat{\mathbf{a}}_x \qquad \vec{\mathbf{F}}_{gravity} = m\,g\,\widehat{\mathbf{n}}_z$$

- **Draw** the aircraft's ***free-body diagram (FBD)*** and form the net (resultant) force on the aircraft.

Result: $\vec{\mathbf{F}}_{Net} = (F_T - F_D)\,\widehat{\mathbf{a}}_x - F_L\,\widehat{\mathbf{a}}_z + m\,g\,\widehat{\mathbf{n}}_z$

- Knowing each point of the aircraft moves with **constant** velocity $v\,\widehat{\mathbf{n}}_x$ (constant-speed, constant angle of attack, i.e., the acceleration of each point is $\vec{\mathbf{0}}$), express F_L and F_D in terms of $m\,g$, F_T, and θ.

Result:

$$\overbrace{(F_T - F_D)\,\widehat{\mathbf{a}}_x - F_L\,\widehat{\mathbf{a}}_z + m\,g\,\widehat{\mathbf{n}}_z}^{\vec{\mathbf{F}}_{Net}} = m\,\overset{\vec{0}}{\vec{\mathbf{a}}} \quad \Rightarrow \quad F_L = m\,g\,\cos(\theta) \qquad F_D = F_T - m\,g\,\sin(\theta)$$

One **model** for the lift and drag associated with the replaced forces on A is[3]

$$F_L = \tfrac{1}{2}\,\rho_{air}\,\text{Area}\,(2\,\pi\,\theta)\,v^2$$
$$F_D = b\,v^2$$

ρ_{air}	$= 1\ \text{kg/m}^3$	Local air density
Area	$= 24\ \text{m}^2$	Surface area of A's bottom surface
b	$= 0.5\ \text{N s}^2/\text{m}^2$	Experimentally-determined drag constant

- Using $F_T = 2200$ N and $m\,g = 9800$ N, form two equations which when solved determine v and θ.

Result: (only in terms of v, θ, π and **numbers** – no calculators necessary). **Optional:** Solve for v and θ.

Equation 1: $9800\,\cos(\theta) - 24\,\pi\,\theta\,v^2 = 0$

Equation 2: $2200 - \tfrac{1}{2}\,v^2 - 9800\,\sin(\theta) = 0$

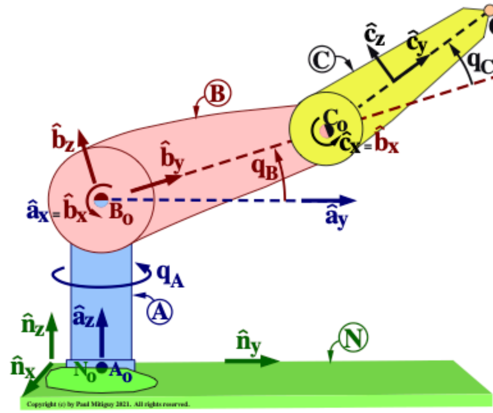
$v \approx$	$61\ \frac{\text{m}}{\text{s}}$	$\approx$	136 mph
$\theta \approx$	2°	$\approx$	0.035 rad

[2]For simplicity, the aircraft's center of mass A_{cm} is coincident with the aircraft's "***aerodynamic center***".

[3]This **model** uses simplifying assumptions (including zero velocity free-stream air, thin airfoil theory, incompressible flow, small angles of attack, etc.) that result in a ***lift coefficient*** $C_L = 2\,\pi\,\alpha$ (α is A's ***angle of attack***, grossly approximated as $\alpha \approx \theta$). Aerodynamic drag forces are a function of Mach number, geometry, angle of attack, etc., and act on each point of the aircraft in a direction opposing the motion of the point in the free-stream air.

4.17 Rotation matrices for a welding robot. (Section 5.5).

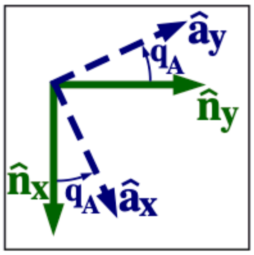
The figure to the right shows a welding robot with hub A and rigid links B and C. There are four sets of orthogonal right-handed unit vectors, namely:



$\hat{n}_x, \hat{n}_y, \hat{n}_z$ fixed to Earth N	$\hat{b}_x, \hat{b}_y, \hat{b}_z$ fixed to link B
$\hat{a}_x, \hat{a}_y, \hat{a}_z$ fixed to hub A	$\hat{c}_x, \hat{c}_y, \hat{c}_z$ fixed to link C

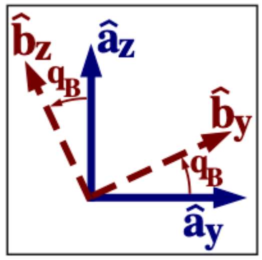
- Redraw $\hat{a}_x$, $\hat{a}_y$ and $\hat{n}_x$, $\hat{n}_y$ in a geometrically suggestive way to form the ${}^aR^n$ ***rotation table***. Also form ${}^N\vec{\omega}^A$ (A's angular velocity in N). Similarly for the rotation tables ${}^bR^a$, ${}^cR^b$ and angular velocities ${}^A\vec{\omega}^B$, ${}^B\vec{\omega}^C$.

Result:



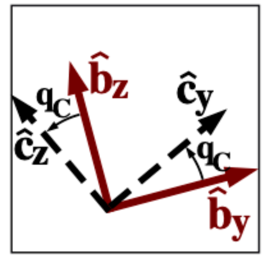
${}^aR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$	$\cos(q_A)$	$\sin(q_A)$	0
$\hat{a}_y$	$-\sin(q_A)$	$\cos(q_A)$	0
$\hat{a}_z$	0	0	1

$${}^N\vec{\omega}^A = \dot{q}_A\,\hat{a}_z$$



${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$	1	0	0
$\hat{b}_y$	0	$\cos(q_B)$	$\sin(q_B)$
$\hat{b}_z$	0	$-\sin(q_B)$	$\cos(q_B)$

$${}^A\vec{\omega}^B = \dot{q}_B\,\hat{b}_x$$



${}^cR^b$	$\hat{b}_x$	$\hat{b}_y$	$\hat{b}_z$
$\hat{c}_x$	1	0	0
$\hat{c}_y$	0	$\cos(q_C)$	$\sin(q_C)$
$\hat{c}_z$	0	$-\sin(q_C)$	$\cos(q_C)$

$${}^B\vec{\omega}^C = \dot{q}_C\,\hat{c}_x$$

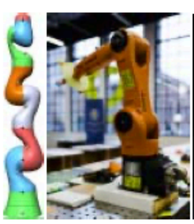
- Form the ${}^cR^a$ rotation table and ${}^N\vec{\omega}^C$ (C's angular velocity in N).

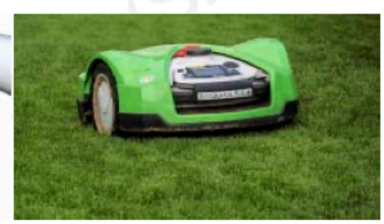
Result:

${}^cR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{c}_x$	1	0	0
$\hat{c}_y$	0	$\cos(q_B+q_C)$	$\sin(q_B+q_C)$
$\hat{c}_z$	0	$-\sin(q_B+q_C)$	$\cos(q_B+q_C)$

Angular velocity addition theorem.

$${}^N\vec{\omega}^C = {}^N\vec{\omega}^A + {}^A\vec{\omega}^B + {}^B\vec{\omega}^C = \dot{q}_A\,\hat{a}_z + \dot{q}_B\,\hat{b}_x + \dot{q}_C\,\hat{c}_x$$





4.18 ♣ Efficient calculation of the inverse of a rotation matrix. (Section 5.4.2).

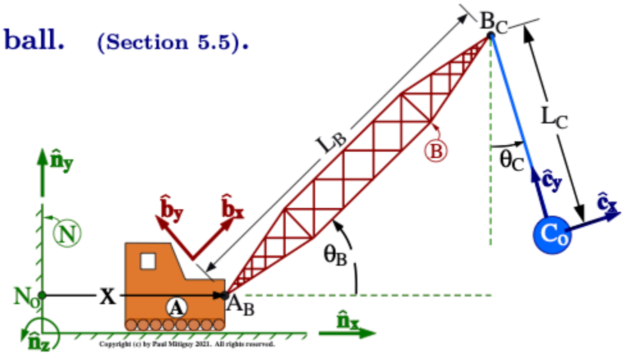
The following rotation matrix R relates two right-handed, orthogonal, unitary bases.
Calculate its inverse by-hand (no calculator) in less than 30 seconds.

$$R = \begin{bmatrix} 0.3830 & -0.6634 & 0.6428 \\ 0.9237 & 0.2795 & -0.2620 \\ -0.0058 & 0.6941 & 0.7198 \end{bmatrix} \quad \Rightarrow \quad R^{-1} = \begin{bmatrix} 0.3830 & 0.9237 & -0.0058 \\ -0.6634 & 0.2795 & 0.6941 \\ 0.6428 & -0.2620 & 0.7198 \end{bmatrix}$$

4.19 Rotation matrices for a crane and wrecking ball. (Section 5.5).

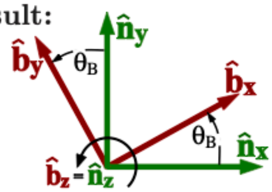
The figure to the right shows a crane whose cab A supports a boom B that swings a wrecking ball C. There are three sets of mutually perpendicular right-handed unit vectors, namely $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$; $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$; and $\hat{c}_x$, $\hat{c}_y$, $\hat{c}_z$. The point of this problem is to relate these sets of unit vectors.

Note: To relate $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ to $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$, these vectors are redrawn in the geometrically suggestive way shown below.



- Use sine, cosine, and vector addition to express each of $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ in terms of $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$. Use these expressions to form the ${}^bR^n$ ***rotation table***. Also form ${}^N\vec{\omega}^B$ (B's angular velocity in N).

Result:



$$\hat{b}_x = \cos(\theta_B)\,\hat{n}_x + \sin(\theta_B)\,\hat{n}_y$$
$$\hat{b}_y = -\sin(\theta_B)\,\hat{n}_x + \cos(\theta_B)\,\hat{n}_y$$
$$\hat{b}_z = \hat{n}_z$$
$${}^N\vec{\omega}^B = \dot{\theta}_B\,\hat{n}_z$$

${}^bR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{b}_x$	$\cos(\theta_B)$	$\sin(\theta_B)$	0
$\hat{b}_y$	$-\sin(\theta_B)$	$\cos(\theta_B)$	0
$\hat{b}_z$	0	0	1

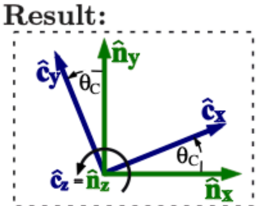
- Form ${}^bR^n$, the ***rotation matrix*** relating $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ to $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$. Then form its transpose ${}^nR^b$.

Result:

$$\begin{bmatrix} \hat{b}_x \\ \hat{b}_y \\ \hat{b}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_B) & \sin(\theta_B) & 0 \\ -\sin(\theta_B) & \cos(\theta_B) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix} \qquad \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_B) & -\sin(\theta_B) & 0 \\ \sin(\theta_B) & \cos(\theta_B) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_x \\ \hat{b}_y \\ \hat{b}_z \end{bmatrix}$$

- To relate $\hat{c}_x$, $\hat{c}_y$, $\hat{c}_z$ to $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$, redraw these unit vectors in a geometrically suggestive way and then use vector addition and sine and cosine to express each of $\hat{c}_x$, $\hat{c}_y$, $\hat{c}_z$ in terms of $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$. Use these expressions to form the ${}^cR^n$ ***rotation table***. Also form C's angular velocity in N.

Result:



$$\hat{c}_x = \cos(\theta_C)\,\hat{n}_x + \sin(\theta_C)\,\hat{n}_y$$
$$\hat{c}_y = -\sin(\theta_C)\,\hat{n}_x + \cos(\theta_C)\,\hat{n}_y$$
$$\hat{c}_z = \hat{n}_z$$

${}^cR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{c}_x$	$\cos(\theta_C)$	$\sin(\theta_C)$	0
$\hat{c}_y$	$-\sin(\theta_C)$	$\cos(\theta_C)$	0
$\hat{c}_z$	0	0	1

$${}^N\vec{\omega}^C = \dot{\theta}_C\,\hat{n}_z$$

- Form the ${}^bR^c$ rotation table via matrix multiplication ${}^bR^c = {}^bR^n * {}^nR^c$.

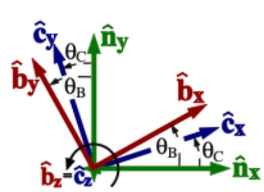
Simplify the results with the following trigonometric identities (repeated from Section 1.4.2).

$\sin(\theta_B+\theta_C) = \sin(\theta_B)\cos(\theta_C) + \sin(\theta_C)\cos(\theta_B)$	$\sin(\theta_B-\theta_C) = \sin(\theta_B)\cos(\theta_C) - \sin(\theta_C)\cos(\theta_B)$
$\cos(\theta_B+\theta_C) = \cos(\theta_B)\cos(\theta_C) - \sin(\theta_B)\sin(\theta_C)$	$\cos(\theta_B-\theta_C) = \cos(\theta_B)\cos(\theta_C) + \sin(\theta_B)\sin(\theta_C)$

Use the redrawn $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ and $\hat{c}_x$, $\hat{c}_y$, $\hat{c}_z$ to visualize results. Form C's angular velocity in B.

Result:

${}^bR^c$	$\hat{c}_x$	$\hat{c}_y$	$\hat{c}_z$
$\hat{b}_x$	$\cos(\theta_B-\theta_C)$	$\sin(\theta_B-\theta_C)$	0
$\hat{b}_y$	$-\sin(\theta_B-\theta_C)$	$\cos(\theta_B-\theta_C)$	0
$\hat{b}_z$	0	0	1



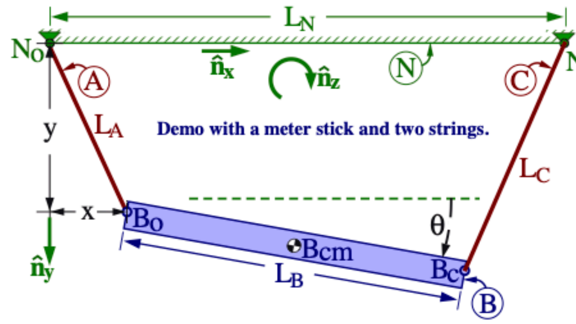
Angular velocity addition theorem.

$$\begin{aligned} {}^B\vec{\omega}^C &= {}^B\vec{\omega}^N + {}^N\vec{\omega}^C \\ &= -{}^N\vec{\omega}^B + {}^N\vec{\omega}^C \\ &= -\dot{\theta}_B\,\hat{n}_z + \dot{\theta}_C\,\hat{n}_z \\ &= (\dot{\theta}_C - \dot{\theta}_B)\,\hat{n}_z \end{aligned}$$

4.20 Optional: Specifying cable lengths to position a construction hoist (Sections 5.5 and 2.9).

A uniform beam B is attached to a roof N by two variable-length cables (A and C).
Cable A attaches to the roof at point N_o of N and to the beam at point B_o of B.
Cable C attaches to the roof at point N_C of N and to the beam at point B_C of B.
Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in N and B, with:

- $\widehat{\mathbf{n}}_x$ horizontally-right from N_o to N_C
- $\widehat{\mathbf{n}}_y$ vertically-**downward**
- $\widehat{\mathbf{n}}_z = \widehat{\mathbf{b}}_z$ inward normal to the vertical plane containing points N_o, B_o, B_C, N_C.
- $\widehat{\mathbf{b}}_x$ directed from B_o to B_C.



Description	Symbol	Value
Distance between N_o and N_C	L_N	6 m
Distance between B_o and B_C	L_B	4 m
Length of cable A	L_A	2.7 m
Length of cable C	L_C	3.7 m
$\widehat{\mathbf{n}}_x$ measure of position from N_o to B_o	x	Variable
$\widehat{\mathbf{n}}_y$ measure of position from N_o to B_o	y	Variable
Angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{b}}_x$ with $+\widehat{\mathbf{n}}_z$ sense	θ	Variable

${}^bR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

$${}^N\vec{\boldsymbol{\omega}}^B = \dot{\theta}\,\widehat{\mathbf{b}}_z$$

Although planar geometry and the Pythagorean theorem relate the cables' lengths to x, y, θ, these techniques are less effective than vector methods for complicated or 3D geometry.

(a) **Draw** $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$, complete the previous ${}^bR^n$ rotation table. Form ${}^N\vec{\boldsymbol{\omega}}^B$ (B's angular velocity in N).

(b) Using **only** the picture,[4] complete the following blanks in terms of x, y, L_B, L_N.
Result:

Position from N_o to B_o $\quad {}^{N_o}\vec{\mathbf{r}}^{B_o} = x\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y$

Position from N_C to B_C $\quad {}^{N_C}\vec{\mathbf{r}}^{B_C} = (x - L_N)\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y + L_B\,\widehat{\mathbf{b}}_x$

(c) Calculate each cable's ***length*** with a ***dot-products***. Use the following ***distance*** formulas (and the rotation matrix) to **efficiently** relate L_A^2 and L_C^2 to x, y, θ, L_N, L_B.
Result:

$${}^{N_o}\vec{\mathbf{r}}^{B_o} \cdot {}^{N_o}\vec{\mathbf{r}}^{B_o} = L_A^2 = x^2 + y^2$$

$${}^{N_C}\vec{\mathbf{r}}^{B_C} \cdot {}^{N_C}\vec{\mathbf{r}}^{B_C} = L_C^2 = L_B^2 + y^2 + (L_N - x)^2 + 2\,L_B\,y\,\sin(\theta) - 2\,L_B\,\cos(\theta)\,(L_N - x)$$

(d) Knowing $x = 2$ m, determine physically meaning values of y and θ (2^+ significant digits).
Result: (consider using a computer to solve the nonlinear algebraic equations).

$$y = 1.814 \text{ m} \qquad \theta = 27.674°$$

(e) **Implicitly** differentiate L_A^2 and L_C^2 to efficiently relate $\dot{L}_A$ and $\dot{L}_C$ to $\dot{x}$, $\dot{y}$, $\dot{\theta}$.
Result: (for this problem only, regard L_A and L_C as non-constant, i.e., $L_A(t)$ and $L_C(t)$ are functions of time).

$$2\,L_A\,\dot{L}_A = 2\,(x\,\dot{x} + y\,\dot{y})$$

$$2\,L_C\,\dot{L}_C = 2\left\{[y + L_B\,\sin(\theta)]\,\dot{y} + L_B\,[y\,\cos(\theta) + \sin(\theta)\,(L_N - x)]\,\dot{\theta} - [L_N - x - L_B\,\cos(\theta)]\,\dot{x}\right\}$$

(f) **† Optional:** Calculate x, y, and θ when the beam is in static equilibrium.
Result: $x \approx 0.815$ m $\quad y \approx 2.574$ m $\quad \theta \approx 12.93°$
Note: Steps (a) and (b) make it easier to complete Homework 4.24.

[4]**Hint**: To form ${}^{N_C}\vec{\mathbf{r}}^{B_C}$, use your finger to trace various paths to B_C from N_C.

4.21 Rotation matrices and angles. (Section 5.5).

Given: Three sets of right-handed orthogonal unitary bases $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$, $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$, $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ and the rotation matrices ${}^aR^c$, ${}^bR^c$, ${}^aR^b$.

${}^aR^c$	$\hat{\mathbf{c}}_x$	$\hat{\mathbf{c}}_y$	$\hat{\mathbf{c}}_z$
$\hat{\mathbf{a}}_x$	0.5	0.866	0
$\hat{\mathbf{a}}_y$	-0.866	0.5	0
$\hat{\mathbf{a}}_z$	0	0	1

${}^bR^c$	$\hat{\mathbf{c}}_x$	$\hat{\mathbf{c}}_y$	$\hat{\mathbf{c}}_z$
$\hat{\mathbf{b}}_x$	$\cos(x)\ \cos(y)$	$\sin(x)\ \cos(y)$	$-\sin(y)$
$\hat{\mathbf{b}}_y$	$-\sin(x)$	$\cos(x)$	0
$\hat{\mathbf{b}}_z$	$\sin(y)\ \cos(x)$	$\sin(x)\ \sin(y)$	$\cos(y)$

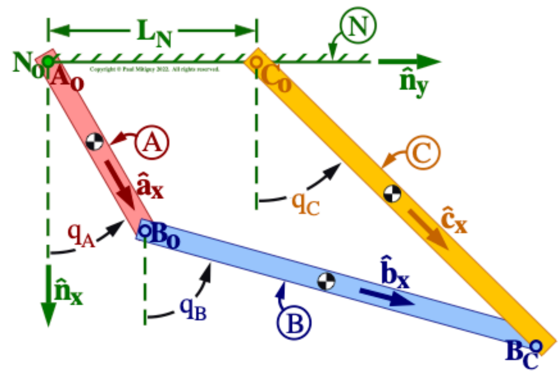
${}^aR^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_x$	$0.5\cos(x)\cos(y) + 0.866\sin(x)\cos(y)$	$0.866\cos(x) - 0.5\sin(x)$	$0.5\sin(y)\cos(x) + 0.866\sin(x)\sin(y)$
$\hat{\mathbf{a}}_y$	$0.5\sin(x)\cos(y) - 0.866\cos(x)\cos(y)$	$0.5\cos(x) + 0.866\sin(x)$	$0.5\sin(x)\sin(y) - 0.866\sin(y)\cos(x)$
$\hat{\mathbf{a}}_z$	$-\sin(y)$	0	$\cos(y)$

Form an expression for the angle between $\hat{\mathbf{a}}_x$ and the vector $\hat{\mathbf{b}}_x + \hat{\mathbf{c}}_x$ in terms of x and y.

Result:

$$\angle(\hat{\mathbf{a}}_x,\ \hat{\mathbf{b}}_x + \hat{\mathbf{c}}_x) \;=\; \text{acos}\left(\frac{0.5 + 0.5\cos(x)\cos(y) + 0.866\sin(x)\cos(y)}{{}^{+}\sqrt{2 + 2\cos(x)\cos(y)}}\right)$$

4.22 Configuration constraints for a four-bar linkage.



A planar four-bar linkage consists of uniform rigid links A, B, C and ground N. Link A is connected with revolute joints to N and B at points N_A and A_B, respectively. Link C is connected with revolute joints to N and B at points C_N and B_C, respectively.

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_i$, $\hat{\mathbf{b}}_i$, $\hat{\mathbf{c}}_i$, $\hat{\mathbf{n}}_i$ $(i = x, y, z)$ are fixed in A, B, C, N, with $\hat{\mathbf{a}}_x$ directed from N_A to A_B, $\hat{\mathbf{b}}_x$ from A_B to B_C, $\hat{\mathbf{c}}_x$ from C_N to B_C, $\hat{\mathbf{n}}_x$ vertically-downward, $\hat{\mathbf{n}}_y$ from N_A to C_N, and $\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z = \hat{\mathbf{c}}_z = \hat{\mathbf{n}}_z$ parallel to the revolute joints' axes.

Create a vector "***loop equation***" using a sum of position vectors that start and end at point N_A.

Result:

$$L_A\,\hat{\mathbf{a}}_x \;+\; L_B\,\hat{\mathbf{b}}_x \;+\; -L_C\,\hat{\mathbf{c}}_x \;+\; -L_N\,\hat{\mathbf{n}}_y \;=\; \vec{0}$$

Quantity	Symbol	Value
Distance from N_A to A_B	L_A	1 m
Distance from A_B to B_C	L_B	2 m
Distance from B_C to C_N	L_C	2 m
Distance from C_N to N_A	L_N	1 m
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$	q_A	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{b}}_x$	q_B	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{c}}_x$	q_C	Variable

Engineering convention: Angles are drawn **positive**.

Dot the loop equation with $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$ to create two equations $f_i = 0$ $(i = 1, 2)$ that relate q_A, q_B, q_C. Next, determine values of q_B and q_C that satisfy these two equations when $q_A = 30°$.

Result: Equations relating q_A, q_B, q_C.	Values when $q_A = 30°$
$f_1 = L_A * \cos(q_A) + L_B * \cos(q_B) - L_C * \cos(q_C)$	q_B = 74.4775°
$f_2 = L_A * \sin(q_A) + L_B * \sin(q_B) - L_C * \sin(q_C) - L_N$	q_C = 45.5225°

Dot-products can be calculated by definition (inspection of the figure) or with rotation matrices.
If $L_A < 1$ m, link A can be driven completely around, whereas if $L_A > 1$ m, it can only be driven 90°.

Examples of 4-bar linkages: Courtesy Design Simulation Technology (SimWise)

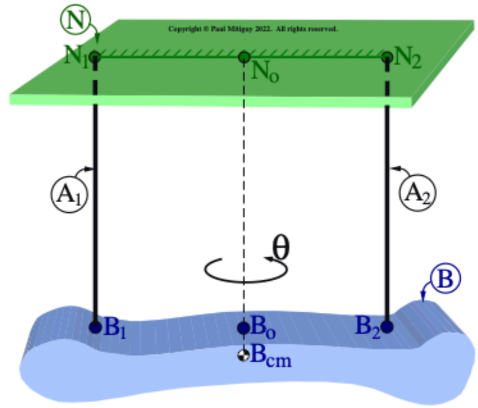
4.23 † Vertical displacement of a bifilar pendulum (useful for calculating moment of inertia).

Bifilar and trifilar pendulum are used to determine inertia properties of rigid bodies (e.g., aircraft, spacecraft, and biological structures such as humans limbs). The following shows a rigid human bone B suspended by two inextensible cables A_1 and A_2, each of which is attached to a flat ceiling N.

- Cable A_1 attaches to the ceiling at point N_1 of N and to the bone at point B_1 of B.
- Cable A_2 attaches to the ceiling at point N_2 of N and to the bone at point B_2 of B.
- Point N_o of N is centered between N_1 and N_2. Point B_o of B is centered between B_1 and B_2.
- Point B_{cm} (B's center of mass) and point B_o ***always*** lie directly below N_o.
- Initially, B_i lies directly below N_i (i=1, 2), respectively.
- B is rotated by an angle θ about the vertical line through B_o and N_o.

- Relate y to L, h, and θ (defined in the following table).

Result: $y^2 \;+\; \frac{1}{2} L^2 \, [1 - \cos(\theta)] \;-\; h^2 \;=\; 0$



- Calculate numerical values for y and $\dot{y}$ (3 significant digits).

Description	Symbol	Value
Distance between N_1 and N_2	L	1 m
Distance between N_i and B_i (i=1, 2)	h	1 m
B's rotation angle in N	θ	135°
B's rotation rate in N	$\dot{\theta}$	0.5 $\frac{\text{rad}}{\text{sec}}$
Distance between N_o and B_o	y	0.383 m
Time-derivative of y	$\dot{y}$	-0.231 $\frac{\text{m}}{\text{s}}$

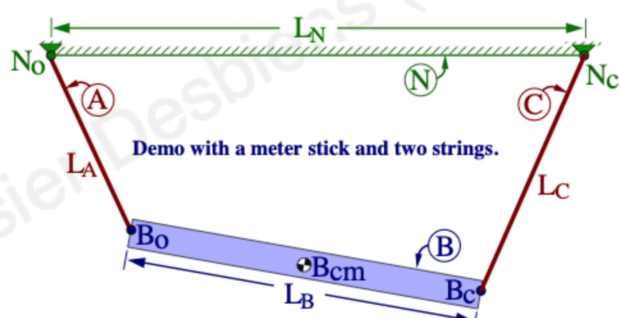
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ 2D/3D geometry.

Courtesy Doug Schwandt

4.24 † Beam location (if you are unable to do this, first try Homeworks 2.20, 4.20).

A uniform beam B is attached to a roof N by two cables (A and C).
Cable A attaches to the roof at point N_o of N and to the beam at point B_o of B.
Cable C attaches to the roof at point N_C of N and to the beam at point B_C of B.
N_o, B_o, B_{cm}, B_C, N_C are all in the same vertical plane.

Description	Symbol	Type	Value
Distance between N_o and N_C	L_N	Constant	6 m
Distance between B_o and B_C	L_B	Constant	4 m
Length of cable A	L_A	Constant	2.7 m
Length of cable C	L_C	Constant	3.7 m



Calculate the distance between N_o and B_{cm} when the beam is stationary in N.

Result (4 significant digits): distance ≈ 4.0955 m

Instructor: See www.MotionGenesis.com ⇒ Get Started ⇒ Statics

Verify your intuition/analysis predicts vertical cables and a horizontal beam for the special case $L_B = L_N$ and $L_C = L_A$.