Homework 5. Chapters 1, 7.
Calculus review and vector differentiation

Show work – except for ♣ fill-in-blanks.

5.1 ♣ Notations for derivatives (complete the blanks). (Section 1.6.1).

Symbol for	1^{st},	2^{nd},	3^{rd} derivative	Idea	Date	Name of mathematician
Compact	$\dot{y}$	$\ddot{y}$	$\dddot{y}$	**Geometry/slope**	1675	Newton
Explicit	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$	**Differentials**	1675	Leibniz (taught Bernoullis who tutored **Euler**)
Keyboard	y'	y''	y'''	**Functions**	1797	Euler and Lagrange (who was trained by **Euler**)
	$\lim\limits_{h \to 0} \frac{y(t+h) - y(t)}{h}$	?	?	**Limits** **delta-epsilon**	1850 1872	Cauchy (trained by **Lagrange**) Weierstrass
	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$		1786 1841	Legendre (introduced partials, abandoned them) Jacobi (re-introduced partials again)

There was bitter rivalry between Newton and Leibniz about the concepts and notation for a derivative.

5.2 ♣ (1675 AD) Leibniz's shorthand notation for 3^{rd} derivatives. (Section 1.6.1).

Write the explicit expression for Leibniz's 3^{rd} derivative show right (so it contains three 1^{st} derivatives).

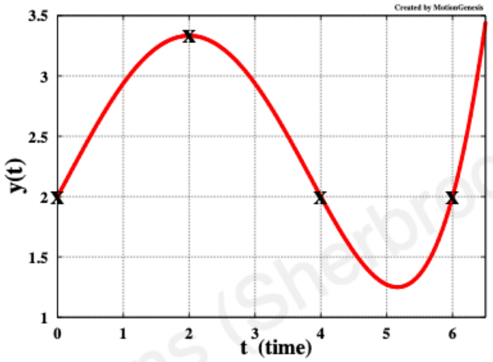
$$\underbrace{\frac{d^3y}{dt^3}}_{\text{shorthand}} \triangleq \underbrace{\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{dy}{dt}\right)\right)}_{\text{explicit}}$$

Write Leibniz's and Newton's shorthand expression for the 9^{th} derivative of y with respect to t.

$$\underbrace{\frac{d^9y}{dt^9}}_{\text{Leibniz}} = \underbrace{\overset{\vdots\vdots\vdots}{\mathbf{y}}}_{\text{Newton}}$$

5.3 ♣ (1675 AD) Newton's idea: Derivative as geometry (slope and curvature). (Section 1.6.1).

Newton related derivatives to geometry (1^{st}-derivative as slope and 2^{nd}-derivative as curvature). Estimate the slope of the function $y(t)$ shown right at $t = 0,\ 2,\ 4,\ 6$.



Result: Pick your answers from: $\boxed{\textbf{-1},\ \textbf{0},\ \textbf{1},\ \textbf{2}}$.

Slope (1^{st} **derivative**)

$$\left.\frac{dy}{dt}\right|_{t=0} \approx 1 \qquad \left.\frac{dy}{dt}\right|_{t=2} \approx 0$$

$$\left.\frac{dy}{dt}\right|_{t=4} \approx -1 \qquad \left.\frac{dy}{dt}\right|_{t=6} \approx 2$$

Estimate the **sign** of the curvature [2^{nd}-derivative of $y(t)$].

Result: Pick your answers from: $\boxed{<,\ \approx,\ >}$. Select $\approx$ when the curvature ≈ 0 (i.e., $|\frac{d^2y}{dx^2}| < 0.01$).

Curvature (2^{nd} **derivative**)

$$\left.\frac{d^2y}{dt^2}\right|_{t=0} \approx 0 \qquad \left.\frac{d^2y}{dt^2}\right|_{t=2} < 0 \qquad \left.\frac{d^2y}{dt^2}\right|_{t=4} \approx 0 \qquad \left.\frac{d^2y}{dt^2}\right|_{t=6} > 0$$

5.4 ♣ (1755 AD) Euler's idea: Derivative of a function is a function. (Section 1.6.5).

Differentiate the following functions that depend on t (time). Express results in terms of x, $\dot{x}$, t so the results are valid when x is constant or depends on time (e.g., when $x = 9$ or $x = t^3$ or $x = t^5$ or ...).

Result:

$$\begin{array}{lll}
\frac{d}{dt}\, t^2 = 2\,t & \frac{d}{dt}\, t^3 = 3\,t^2 & \frac{d}{dt}\, t^{-7} = -7\,t^{-8} \\
\frac{d}{dt}\, \sin(t) = \cos(t) & \frac{d}{dt}\, \cos(t) = -\sin(t) & \frac{d}{dt}\, \cos(x) = -\sin(x) * \dot{x} \\
\frac{d}{dt}\, e^t = e^t & \frac{d}{dt}\, \ln(t) = \frac{1}{t} & \frac{d}{dt}\, \ln(x) = \frac{1}{x} * \dot{x}
\end{array}$$

5.5 ♣ Good product rule for differentiation – for scalars, $\vec{\text{v}}$ectors, [matrices], ... (Section 1.6.7).

Circle the ***good product rule*** that works when u and v are scalars or $\vec{\text{v}}$ectors, or u is a **2 × 3** matrix and v is a **3 × 5** matrix (if you did not learn the **good product rule**, update your calculus teacher).

$$\boxed{\frac{d(u * v)}{dt} = \frac{du}{dt} * \text{v} + \text{u} * \frac{dv}{dt}} \qquad \frac{d(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \qquad \frac{d(u * v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

Knowing u, v, w are scalars or matrices that depend on time t, use the ***good product rule for differentiation*** to form the

Good product rule: $\dfrac{dy}{dt} = \dfrac{d(u * v * w)}{dt} = \dfrac{du}{dt}\, v\, w \;+\; u\, \dfrac{dv}{dt}\, w \;+\; u\, v\, \dfrac{dw}{dt}$

5.6 ♣ Example of the "good product rule" for differentiation (if done right, takes ≈ 2 minutes).

Differentiate the function $f(t)$ with the easy-to-use ***good product rule for differentiation***.

$$\begin{array}{llcccccccccc}
\textbf{Function:} & f(t) & = & \sin(t) & * & \cos(t) & * & t^2 & * & e^t & * & \ln(t) \\
\hline
\textbf{Derivative:} & \frac{df}{dt} & = & \cos(t) & * & \cos(t) & * & t^2 & * & e^t & * & \ln(t) \\
 & & + & \sin(t) & * & -\sin(t) & * & t^2 & * & e^t & * & \ln(t) \\
 & & + & \sin(t) & * & \cos(t) & * & 2 * t & * & e^t & * & \ln(t) \\
 & & + & \sin(t) & * & \cos(t) & * & t^2 & * & e^t & * & \ln(t) \\
 & & + & \sin(t) & * & \cos(t) & * & t^2 & * & e^t & * & \frac{1}{t}
\end{array}$$

Hint: The "**good product rule**" is an ***efficient*** way to differentiate expressions with many factors.

5.7 ♣ Alternative to quotient rule: combine product/exponent rules. (Section 1.6.8).

Although the ***quotient rule*** can be used to differentiate the ratio of functions $f(t)$ and $g(t)$, it can be easier to remember $\dfrac{f(t)}{g(t)} = f(t) * g(t)^{-1}$ and then use the ***product rule*** as shown below.

Given example:	$\dfrac{\sin(t)}{t} = \sin(t) * t^{-1}$	$\dfrac{d}{dt}\,[\sin(t) * t^{-1}] = \cos(t)\, t^{-1} - \sin(t)\, t^{-2}$
Complete this:	$\dfrac{\sin(t)}{t^2} = \sin(t) * t^{-2}$	$\dfrac{d}{dt}\,[\sin(t) * t^{-2}] = \cos(t)\, t^{-2} - 2\sin(t)\, t^{-3}$

5.8 ♣ Chain rule for differentiation. $\frac{d\,f[x(t)]}{dt} = \frac{df}{dx}\frac{dx}{dt}$ $\quad \frac{d\,f[x,y]}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ (Sections 1.6.9, 1.6.4).

Differentiate the function $f(t)$ with the **chain rule** [$x(t)$ and $y(t)$ depend on the independent variable t (time)].

Function: $f(t) = \sin(x) + y^2 + (\dot{x})^2 + e^x + \ln(y) + \frac{1}{x} + \cos(x+y)$

Derivative: $\dfrac{df}{dt} = \cos(x)\,\dot{x} + 2\,y\,\dot{y} + 2\,\dot{x}\,\ddot{x} + e^x\,\dot{x} + \frac{1}{y}\,\dot{y} - \frac{1}{x^2}\,\dot{x} - \sin(x+y)\,(\dot{x} + \dot{y})$

5.9 ♣ Ordinary derivative of the function $f(t) = \sin(t) * \cos(x\,y\,z)$. (Sections 1.6.7 and 1.6.9).

Differentiate the function $f(t)$ with respect to t [$x(t)$, $y(t)$, $z(t)$ depend on the independent variable t (time)].

Result: $\dfrac{d\,[\sin(t)\,\cos(x\,y\,z)]}{dt} = \cos(t)\,\cos(x\,y\,z) - \sin(t)\,\sin(x\,y\,z)\,(\dot{x}\,y\,z + x\,\dot{y}\,z + x\,y\,\dot{z})$

5.10 ♣ Differentiation concepts. (Section 1.6.10 – implicit differentiation).

$$y^4 \;-\; 8\,y \;=\; 3\,t^2 \;+\; \sin(t)$$

Roots: $y = \boxed{0}$, $y = \boxed{2}$, $y \approx$ -1 $\pm$ $1.732\,i$

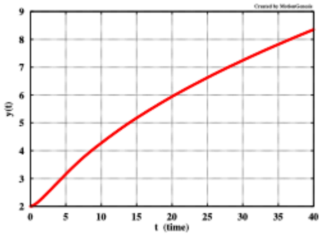
The equation to the right relates the dependent variable $y(t)$ to the independent variable t. Find two real roots to this equation when $t = 0$.

Form a general expression for $\frac{dy}{dt}$ in terms of y and t and calculate $\frac{dy}{dt}$ when $t = 0$ and $y = 2$.

Result:
In terms of t and y: $\frac{dy}{dt} = \frac{6\,t \;+\; \cos(t)}{4\,y^3 \;-\; 8}$ Numerical value: $\left.\frac{dy}{dt}\right|_{\substack{t=0\\y=2}} = \frac{1}{24}$

†Optional: Continuous solution of nonlinear algebraic equation.

Starting with $y = 2$, continuously solve for $y(t)$ for $0 \le t \le 40$ and plot your results as shown right. Stumped: See hint in Homework 5.32.

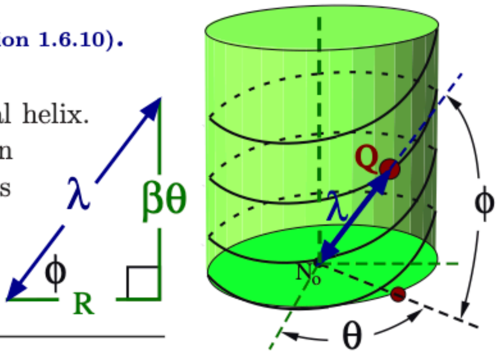


5.11 ♣ Review of explicit and implicit differentiation. (Section 1.6.10).

The figure to the right shows a point Q sliding on a cylindrical helix. Two geometrically significant variables are a distance λ and an angle ϕ that are related to **constants** R, β and a variable θ as

$$\lambda^2 \;=\; R^2 \;+\; (\beta\,\theta)^2 \qquad\qquad \tan(\phi) \;=\; \frac{\beta\,\theta}{R}$$

Form $\dot\lambda$ and $\dot\phi$ using the two methods described below.



Explicit differentiation
1. Solve explicitly for λ and ϕ.
2. Then differentiate the resulting expressions.

Result:
In terms of R, β, θ, $\dot\theta$.

$$\lambda \;=\; \sqrt{R^2 \,+\, (\beta\,\theta)^2} \qquad\qquad \phi \;=\; \text{atan}\big(\tfrac{\beta\,\theta}{R}\big) \quad \text{Hint: } \frac{\partial\,\text{atan}(x)}{\partial x} = \frac{1}{1+x^2}$$

$$\dot\lambda \;=\; \frac{\beta^2\,\theta}{\sqrt{R^2+(\beta\,\theta)^2}}\,\dot\theta \qquad\qquad \dot\phi \;=\; \frac{\beta/R}{1 \,+\, (\frac{\beta\,\theta}{R})^2}\,\dot\theta$$

Implicit differentiation
1. Differentiate the equations for λ^2 and $\tan(\phi)$.
2. Then solve for $\dot\lambda$ and $\dot\phi$.

Result:
In terms of R, β, θ, $\dot\theta$, λ.

$$\dot\lambda \;=\; \frac{\beta^2\,\theta}{\lambda}\,\dot\theta \qquad \dot\phi \;=\; \cos^2(\phi)\;\frac{\beta}{R}\,\dot\theta \;=\; \frac{\beta\,R}{\lambda^2}\,\dot\theta$$

Forming $\dot\lambda$ is easier and computationally more efficient with **explicit**/**implicit** differentiation.

5.12 ♣ Review of partial and ordinary differentiation. (Section 1.6.2).

The mathematical process below is used in ***Lagrange's equations of motion***.

The kinetic energy K of a bridge-crane (shown right) can be written in terms of constants M, m, L and variables x, $\dot x$, θ, $\dot\theta$, as

$$\text{K} \;=\; \tfrac{1}{2}\,M\,\dot x^2 \;\;+\;\; \tfrac{1}{2}\,m\,L\,[L\,\dot\theta^2 + 2\,\cos(\theta)\,\dot x\,\dot\theta]$$

- First, regard x, $\dot x$, θ, $\dot\theta$ as independent variables [so K depends on each separately, i.e., $\text{K}(x,\,\dot x,\,\theta,\,\dot\theta)$], form the **partial derivatives** below (left).
- Next, regard x, $\dot x$, θ, $\dot\theta$ as time-dependent variables and form the **ordinary derivatives** below (right).

$$\frac{\partial\text{K}}{\partial\theta} = \text{-}m\,L\,\sin(\theta)\,\dot x\,\dot\theta \qquad \frac{\partial\text{K}}{\partial\dot\theta} = m\,L\,[L\,\dot\theta + \cos(\theta)\,\dot x] \qquad \frac{d}{dt}\left(\frac{\partial\text{K}}{\partial\dot\theta}\right) = m\,L\,[L\,\ddot\theta - \sin(\theta)\,\dot\theta\,\dot x + \cos(\theta)\,\ddot x]$$

$$\frac{\partial\text{K}}{\partial x} = 0 \qquad \frac{\partial\text{K}}{\partial\dot x} = M\,\dot x \;+\; m\,L\,\cos(\theta)\,\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial\text{K}}{\partial\dot x}\right) = M\,\ddot x \;+\; m\,L\,[\text{-}\sin(\theta)\,\dot\theta^2 \;+\; \cos(\theta)\,\ddot\theta]$$

5.13 ♣ Differentiation concepts – what is wrong? (Section 1.6.3 and previous problem).

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ only when the ball reaches maximum height, explain what is wrong with the following statement about v's time derivative.

$\frac{dv}{dt} = \frac{d\,(0)}{dt} = 0$ is **wrong**. We know the correct answer is: $\frac{dv}{dt} = \text{-}g \approx \text{-}9.8\ \frac{\text{m}}{\text{s}^2}$.

Explain what is wrong: It is incorrect to time-differentiate as shown above because: $v = 0$ is an ***instantaneous*** value of v. Time-differentiation must occur over a **non-zero interval** of time. dt is defined as a **non-zero interval** of t (not at an ***instant***).

5.14 ♣ Leibniz's idea and differentiation concepts: What is dt? (Section 1.6.1).

A continuous function $z(t)$ depends on $x(t)$, $y(t)$, and time t as:	$z = x + y^2 \sin(t)$
At a certain instant of time, $y = 1$ and z simplifies to:	$z = x + \sin(t)$

Determine the time-derivative of z at the instant when $y = 1$.

Result: $\left.\frac{dz}{dt}\right|_{y=1} = \dot{x} + 2\,\dot{y}\,\sin(t) + \cos(t)$

5.15 ♣ Euler's idea: Integral of a function is a function. (Section 1.7).

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

Result:

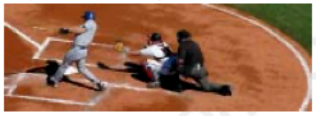
$\int t^2\,dt = \frac{t^3}{3} + C$ $\qquad \int t^3\,dt = \frac{t^4}{4} + C$ $\qquad \int t^8\,dt = \frac{t^9}{9} + C$

$\int t^{-3}\,dt = \frac{t^{-2}}{-2} + C$ $\qquad \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C$ $\qquad \int t^{-1}\,dt = \ln(t) + C$

$\int \sin(t)\,dt = \text{-}\cos(t) + C$ $\qquad \int \cos(t)\,dt = \sin(t) + C$ $\qquad \int e^t\,dt = e^t + C$

$\int 5\,dt = 5\,t + C$ $\qquad \int 5/t\,dt = 5\,\ln(t) + C$ $\qquad \int (5 + \frac{1}{t})\,dt = 5\,t + \ln(t) + C$

5.16 Solve a 1^{st}-order ODE: Separate variables, integrate, initial value. (Section 1.7).



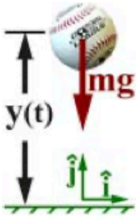
Solve $\frac{dv}{dt} = \text{-}9.8\ \frac{\text{m}}{\text{s}^2}$ with the initial value $v(t{=}0) = 33\ \frac{\text{m}}{\text{s}}$.

Result: $v(t) = \text{-}9.8\,t + 33$ **Show work**

5.17 Solve a 2^{nd}-order ODE: Separate variables, integrate, initial value (twice). (Section 1.7).

Solve $\frac{d^2y}{dt^2} = \text{-}9.8\ \frac{\text{m}}{\text{s}^2}$ with initial values $\dot{y}(t{=}0) = 33\ \frac{\text{m}}{\text{s}}$, $y(t{=}0) = 5$ m. **Show work**

Result: $y(t) = \text{-}4.9\,t^2 + 33\,t + 5$ Hint: $\frac{d^2y}{dt^2} \triangleq \frac{d}{dt}\,(\frac{dy}{dt})$. Separate variables and integrate twice. Use both initial values.



$\boxed{\vec{\mathbf{F}} = \text{m}\,\vec{\mathbf{a}}}$

Physics: Show $\frac{d^2y}{dt^2} = \text{-}9.8\ \frac{\text{m}}{\text{s}^2}$ results from using $\vec{\mathbf{F}} = \text{m}\,\vec{\mathbf{a}}$ for the baseball and simplifying.

Result: $\underbrace{\text{-}m\,g\,\hat{\mathbf{j}}}_{\vec{\mathbf{F}}} = \underbrace{m\,\frac{d^2y}{dt^2}\,\hat{\mathbf{j}}}_{m\,\vec{\mathbf{a}}} \Rightarrow \text{-}m\,g = m\,\frac{d^2y}{dt^2} \Rightarrow \frac{d^2y}{dt^2} = \text{-}g = \text{-}9.8\ \frac{\text{m}}{\text{s}^2}$.

5.18 ♣ Solve a 3^{rd}-order ODE with mixed initial/boundary values. (Section 1.7).

Solve $\frac{d^3y}{dt^3} = 6$ with initial/boundary values $y(t{=}0) = 5$, $\dot{y}(t{=}0) = 0$, $y(t{=}3) = 50$.

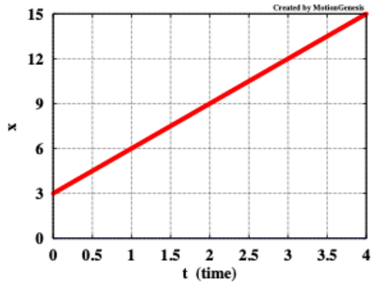
Result: $y(t) = t^3 + 2\,t^2 + 5$ Hint: $\frac{d^3y}{dt^3} \triangleq \frac{d}{dt}\,(\frac{d}{dt}\,(\frac{dy}{dt}))$. Then integrate three times.

5.19 ♣ Geometric interpretations of integrals and derivatives. (Section 1.7).

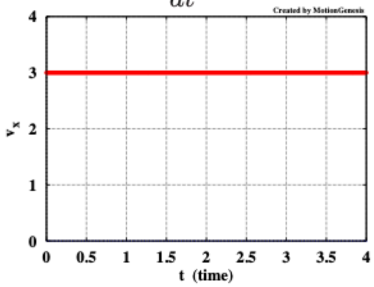
- Complete the blanks and graph the missing functions. Blanks should not have undetermined constants.

Hint: Synthesize information from each vertical column below. Constants of integration can be deduced from graphs. For example, for the 2^{nd} column, start at the bottom with $\frac{d^2x}{dt^2} = 0$ and work upward to determine $\frac{dx}{dt}$ and then $x(t)$.

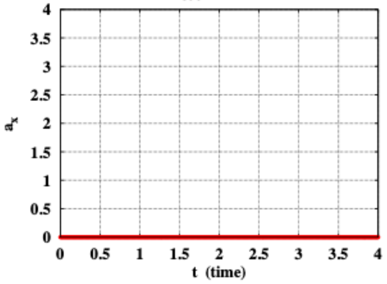
$x = 3t + 3$



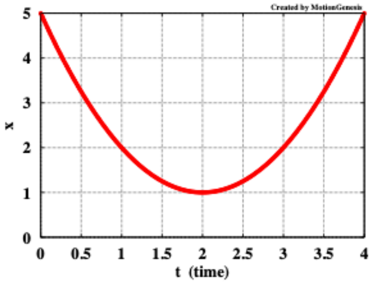
$v_x \triangleq \frac{dx}{dt} = 3$



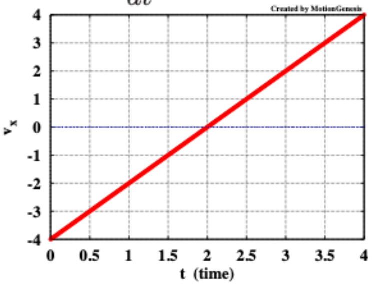
$a_x \triangleq \frac{d^2x}{dt^2} = 0$



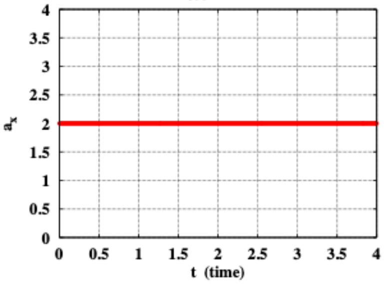
$x = t^2 - 4t + 5$



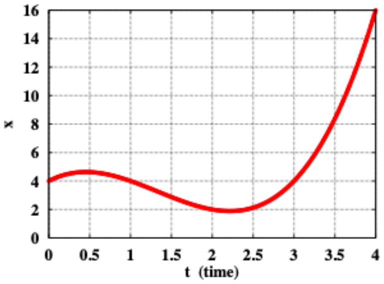
$v_x \triangleq \frac{dx}{dt} = 2t - 4$



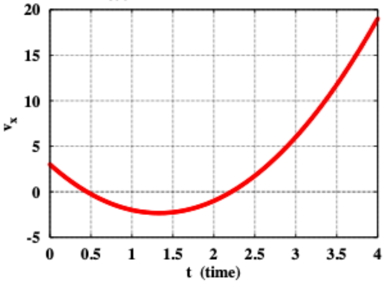
$a_x \triangleq \frac{d^2x}{dt^2} = 2$



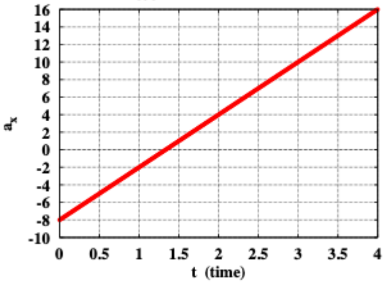
$x = t^3 - 4t^2 + 3t + 4$



$v_x \triangleq \frac{dx}{dt} = 3t^2 - 8t + 3$



$a_x \triangleq \frac{d^2x}{dt^2} = 6t - 8$



$$\boxed{\vec{F} = m\,\vec{a}}$$

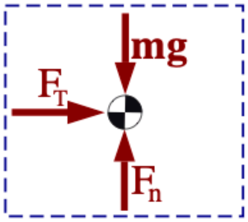
- A rocket-sled/rider is modeled as a particle of mass m whose motion is affected by thrust, normal, and gravity forces. **Draw** its ***free-body diagram*** and write the net force $\vec{F}_{Net}$ in terms of scalars F_T, F_n, $m\,g$ (associated with thrust, normal force, gravity force) and the unit vectors $\hat{i}$ and $\hat{j}$.

Result: $\vec{F}_{Net} = F_T\,\hat{i} + (F_n - m\,g)\,\hat{j}$

- Set $\vec{F}_{Net} = m\,\vec{a}$, form scalar equations, solve for $\ddot{x}$, F_n.

Result:

$$\underbrace{F_T\,\hat{i} + (F_n - m\,g)\,\hat{j}}_{\vec{F}_{Net}} = \underbrace{m\,\ddot{x}\,\hat{i}}_{m\,\vec{a}} \quad \Rightarrow \quad \ddot{x} = \frac{F_T}{m} \qquad F_n = m\,g$$



Thrust $\vec{F}_T = F_T\,\hat{i}$

Normal $\vec{F}_n = F_n\,\hat{j}$

Gravity $\vec{F}_g = -m\,g\,\hat{j}$

$\vec{F}_{Net} = \vec{F}_T + \vec{F}_n + \vec{F}_g$

- Given $m = 100$ kg, $F_T = 800$ Newton, $x(t{=}0) = 7$ m, $\dot{x}(t{=}0) = 0\ \frac{m}{s}$, show $x(t) = 4t^2 + 7$.

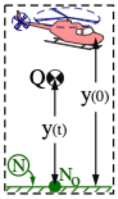
5.20 ♣ FE/EIT: $\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}$ for a sky-diver and rocket-sled. (complete the blanks, graphs, etc).

A sky-diver (modeled as a particle Q of mass m) free-falls for 4 seconds after leaving a stationary helicopter from a height $y(0) = 200$ m above Earth (y is positive-upward).

FBD: Draw Q's ***free-body diagram*** and write the net force on the sky-diver (assume gravity is the only relevant force).

Result: $\vec{\mathbf{F}}_{\text{Net}} = \boxed{-m\,g}\,\widehat{\mathbf{j}}$

Sketch particle Q, Earth's surface N, a point N_o on N, $y(t)$, $y(0)$, and the helicopter. Form $\vec{\mathbf{r}}$, the position vector from N_o to Q. Differentiate $\vec{\mathbf{r}}$ to form Q's velocity $\vec{\mathbf{v}}$ and acceleration $\vec{\mathbf{a}}$.

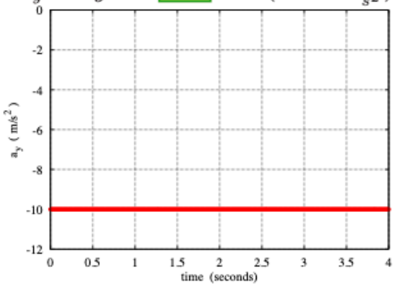


Result: $\vec{\mathbf{r}} = \boxed{y\,\widehat{\mathbf{j}}}$ $\quad\vec{\mathbf{v}} = \boxed{\dot{y}}\,\widehat{\mathbf{j}}$ $\quad\vec{\mathbf{a}} = \boxed{\ddot{y}}\,\widehat{\mathbf{j}}$ in terms of y $\dot{y}$, $\ddot{y}$

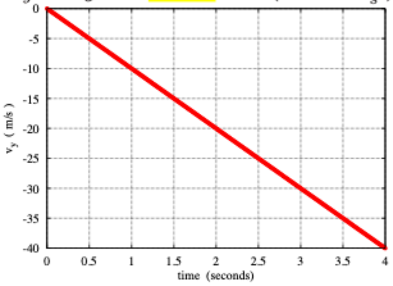
Set $\vec{\mathbf{F}}_{\text{Net}} = \mathrm{m}\,\vec{\mathbf{a}}$, form scalar equation, solve for $\ddot{y}$.

$$\overbrace{\boxed{-m\,g}\,\widehat{\mathbf{j}}}^{\vec{\mathbf{F}}_{\text{Net}}} = \overbrace{\boxed{m\,\ddot{y}}\,\widehat{\mathbf{j}}}^{\mathrm{m}\,\vec{\mathbf{a}}} \quad\Rightarrow\quad \ddot{y} = \boxed{g} \quad (g \approx 10\ \tfrac{\mathrm{m}}{\mathrm{s}^2})$$

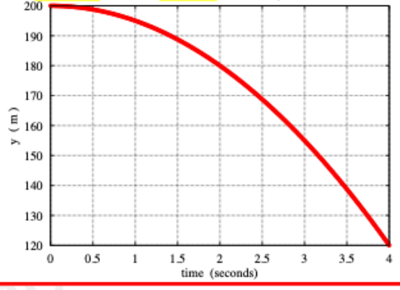
$a_y = \ddot{y} = \boxed{-10}$ (units of $\frac{m}{s^2}$)



$v_y = \dot{y} = \boxed{-10\,t}$ (units of $\frac{m}{s}$)

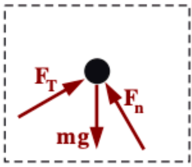


$y = \boxed{-5\,t^2} + \boxed{200}$ (units of meters)



A rocket-sled/rider (modeled as a particle Q of mass m) is thrust along smooth rails with a force F_T.
The variable x measure's the sled's position along the inclined rails.
Initially, $x = 5$ m and $\dot{x} = 0\ \frac{\mathrm{m}}{\mathrm{s}}$.
Unit vector $\widehat{\mathbf{t}}$ is **t**angent to the rails.
Unit vector $\widehat{\mathbf{n}}$ is normal to the rails.

FBD: Draw the forces and write the net force on the rocket-sled/rider.



Result: $\vec{\mathbf{F}}_{\text{Net}} = \boxed{F_T}\,\widehat{\mathbf{t}} + \boxed{F_{\text{n}}}\,\widehat{\mathbf{n}} - \boxed{m\,g}\,\widehat{\mathbf{j}}$

Form Q's position vector, velocity, and acceleration (in terms of x, $\dot{x}$, $\ddot{x}$).

Result: $\vec{\mathbf{r}} = \boxed{x}\,\widehat{\mathbf{t}}$ $\quad\vec{\mathbf{v}} = \boxed{\dot{x}}\,\widehat{\mathbf{t}}$ $\quad\vec{\mathbf{a}} = \boxed{\ddot{x}}\,\widehat{\mathbf{t}}$

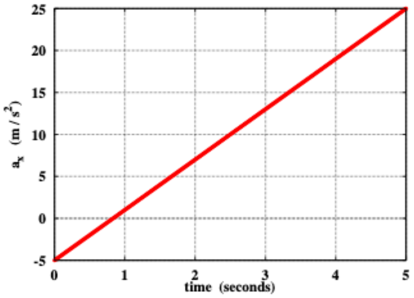
Set $\vec{\mathbf{F}}_{\text{Net}} = \mathrm{m}\,\vec{\mathbf{a}}$, form scalar equations, solve for $\ddot{x}$, F_{n}.

$$\overbrace{[\boxed{F_T} - m\,g\,\boxed{\sin(\theta)}]\,\widehat{\mathbf{t}} + [F_{\text{n}} - m\,g\,\boxed{\cos(\theta)}]\,\widehat{\mathbf{n}}}^{\vec{\mathbf{F}}_{\text{Net}}} = \overbrace{m\,\ddot{x}\,\widehat{\mathbf{t}}}^{\mathrm{m}\,\vec{\mathbf{a}}}$$

$$\ddot{x} = \frac{\boxed{F_T}}{\boxed{m}} - \boxed{g}\,\sin(\theta) \qquad F_{\text{n}} = \boxed{m\,g}\,\cos(\theta)$$

F_{n} measures normal force.

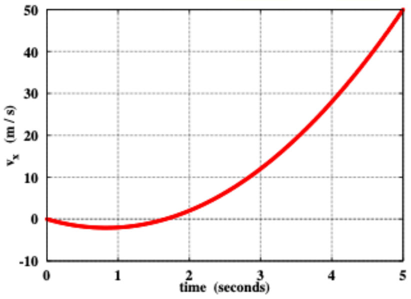
$a_x = \dot{v}_x = \ddot{x} = \boxed{6}\,t - \boxed{5}$ (units of $\frac{m}{s^2}$)



Determine $\ddot{x}$ when:

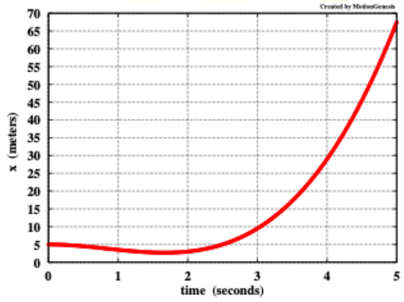
$\theta = 30°$
$m = 100$ kg
$g \approx 10\ \frac{\mathrm{m}}{\mathrm{s}^2}$
$F_T = (600\ \frac{\mathrm{N}}{\mathrm{s}}) * t$.

$v_x = \dot{x} = \boxed{3\,t^2} - \boxed{5\,t}$ (units of $\frac{m}{s}$)



Integrate to form $\dot{x}$. Use the initial value (at time $t = 0$) of $\dot{x}(0) = 0\ \frac{\mathrm{m}}{\mathrm{s}}$ to determine the constant of integration.

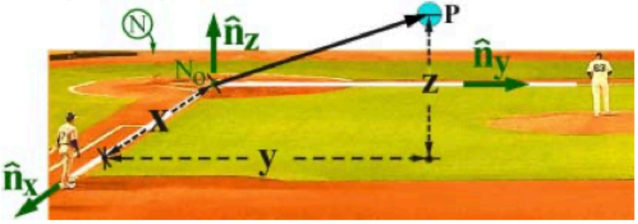
$x = \boxed{t^3} - \boxed{2.5}\,t^2 + \boxed{5}$ (units of meters)



Integrate to form x. Use the initial value (at time $t = 0$) of $x(0) = 5$ m to determine the constant of integration.

5.21 ♣ Vector differentiation: Cartesian coordinates $(x,\, y,\, z)$. (Section 7.1).

The figure to the right shows a baseball P moving over a baseball field (reference frame) N. Orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N with $\widehat{\mathbf{n}}_x$ directed from home-plate to 1^{st}-base, $\widehat{\mathbf{n}}_y$ from home-plate to 3^{rd}-base, and $\widehat{\mathbf{n}}_z$ vertically upward.



$-\infty < x < \infty$ $-\infty < y < \infty$ $-\infty < z < \infty$

- Determine the time-derivative in N of $\widehat{\mathbf{n}}_x$ and justify your answer.

Result: $\dfrac{{}^N d\,\widehat{\mathbf{n}}_x}{dt} = \vec{0}$ The magnitude of $\widehat{\mathbf{n}}_x$ **does/[✓ does not]** change because $|\widehat{\mathbf{n}}_x| = 1$.
The direction of $\widehat{\mathbf{n}}_x$ **does/[✓ does not]** change in reference frame N.

- P's location from point N_o (home-plate) is specified with ***Cartesian coordinates*** x, y, z which depend on time t. By inspection, form P's position from N_o. Using the definition of a vector derivative [equation (7.1)] and the product rule for differentiation, determine the time-derivative in N of $\vec{\mathbf{p}}$.

Result:
$$\vec{\mathbf{p}} = x\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y + z\,\widehat{\mathbf{n}}_z \qquad \frac{{}^N d\vec{\mathbf{p}}}{dt} = \dot{x}\,\widehat{\mathbf{n}}_x + \dot{y}\,\widehat{\mathbf{n}}_y + \dot{z}\,\widehat{\mathbf{n}}_z$$

5.22 ♣ Vector differentiation and reference frames. (Section 7.1).

$$\frac{{}^{\boxed{??}}d\,(\vec{\text{vector}})}{dt}$$

The following vectors are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and t time. Circle the vectors that can be differentiated without consideration of a reference frame or rigid basis.

$\boxed{\vec{0}}$	$2\,\widehat{\mathbf{a}}_x + 4\,\widehat{\mathbf{a}}_y$	$2\,\widehat{\mathbf{a}}_x + t\,\widehat{\mathbf{a}}_y$
$\widehat{\mathbf{a}}_x$	$2\,\widehat{\mathbf{a}}_x + 4\,\widehat{\mathbf{a}}_y + 6\,\widehat{\mathbf{a}}_z$	$2\,\widehat{\mathbf{a}}_x + t\,\widehat{\mathbf{a}}_y + \sin(t)\,\widehat{\mathbf{a}}_z$

5.23 ♣ Textbook/Internet definitions of vector differentiation. (Section 7.1).

$$\frac{{}^{\boxed{??}}d\,(\vec{\text{vector}})}{dt}$$

The derivative of a vector measures its change in ***magnitude*** and its change of ***direction*** in a **reference frame** (or rigid basis). The first notation that explicitly connected a ***vector derivative*** to a ***reference frame*** was by Thomas Kane in 1950. Kane regarded a mathematical ***definition*** to:

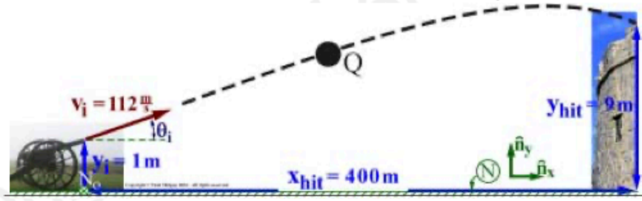
- Involve ingredients that themselves are reasonably understood and/or defined.
  In other words, the definition is comprehensible to the intended audience.
- Be useful for directly or indirectly proving all other related properties.

Report a definition for the derivative of a vector from a textbook (e.g., college/professional math, physics, or engineering textbook) and/or from the Internet, and determine if both the ***definition*** and ***notation*** clearly show that a vector's derivative depends on a **reference frame** (or rigid basis).

Source (reference). List textbook or .html link	Definition. Report the defining equation/property.	Does notation explicitly show dependence on reference frame?
		Yes/No

5.24 FE/EIT – Cannonball velocity $\vec{\mathbf{v}}$, acceleration $\vec{\mathbf{a}}$, and launch angle θ_i. $\boxed{\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}}$

A cannonball Q of mass m is in projectile motion over Earth N. Q's position vector from a point N_o fixed in N is $\vec{\mathbf{r}} = x\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y$ where $\widehat{\mathbf{n}}_x$ is horizontally-right and $\widehat{\mathbf{n}}_y$ is vertically-upward.



- Knowing $\vec{\mathbf{v}} \triangleq \dfrac{{}^N d\vec{\mathbf{r}}}{dt}$ and $\vec{\mathbf{a}} \triangleq \dfrac{{}^N d\vec{\mathbf{v}}}{dt}$, use $\vec{\mathbf{F}}_{\text{Net}} = m\,\vec{\mathbf{a}}$ to form scalar equations and solve for $\ddot{x}$ and $\ddot{y}$.

Result:
$$\overbrace{-m\,g\,\widehat{\mathbf{n}}_y}^{\vec{\mathbf{F}}_{\text{Net}}} = \overbrace{m\,(\ddot{x}\,\widehat{\mathbf{n}}_x + \ddot{y}\,\widehat{\mathbf{n}}_y)}^{m\,\vec{\mathbf{a}}} \;\Rightarrow\; \begin{aligned}\ddot{x} &= 0\\ \ddot{y} &= -g\end{aligned}$$

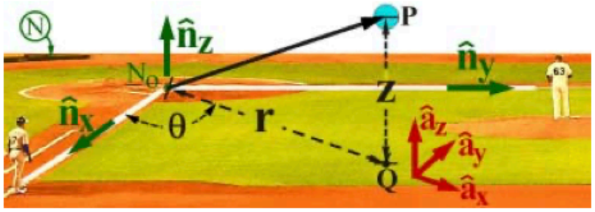
- Q is launched from height $y_i = 1$ m with initial speed $v_i = 112 \frac{\text{m}}{\text{s}}$ to hit a castle wall at $x_{\text{hit}} = 400$ m, $y_{\text{hit}} = 9$ m. Determine θ_i, the angle between Q's initial velocity and $\widehat{\mathbf{n}}_{\text{x}}$ (use $g = 9.8 \frac{\text{m}}{\text{s}^2}$).

Result: $\theta_i \approx \boxed{10.3}°$ †A nonlinear algebraic equation governs θ_i (may require computer solution).

5.25 Cylindrical coordinates (r, θ, z): Orientation and position.

The following figure shows a baseball P moving over a baseball field (reference frame) N. Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$ are fixed in N as shown. P's location from point N_{o} (home-plate) is specified with ***cylindrical coordinates*** r, θ, z which depend on time t.

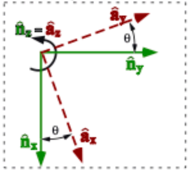
r	the distance between N_{o} and point Q (Q traces P's projection on the flat horizontal playing field).
θ	the angle from $\widehat{\mathbf{n}}_{\text{x}}$ to $\widehat{\mathbf{a}}_{\text{x}}$ with $^+\widehat{\mathbf{n}}_{\text{z}}$ sense. The set A of orthogonal unit vectors $\widehat{\mathbf{a}}_{\text{x}}, \widehat{\mathbf{a}}_{\text{y}}, \widehat{\mathbf{a}}_{\text{z}}$ are directed with $\widehat{\mathbf{a}}_{\text{x}}$ from N_{o} to Q, $\widehat{\mathbf{a}}_{\text{z}} = \widehat{\mathbf{n}}_{\text{z}}$, and $\widehat{\mathbf{a}}_{\text{y}} = \widehat{\mathbf{a}}_{\text{z}} \times \widehat{\mathbf{a}}_{\text{x}}$.
z	the $^+\widehat{\mathbf{n}}_{\text{z}}$ measure of P's position from N_{o}.



$r \geq 0 \qquad -\pi < \theta \leq \pi \qquad -\infty < z < \infty$

(a) **Redraw** $\widehat{\mathbf{a}}_{\text{x}}$, $\widehat{\mathbf{a}}_{\text{y}}$ and $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$ in a geometrically useful way for the $^aR^n$ rotation table.

Result:



$^aR^n$	$\widehat{\mathbf{n}}_{\text{x}}$	$\widehat{\mathbf{n}}_{\text{y}}$	$\widehat{\mathbf{n}}_{\text{z}}$
$\widehat{\mathbf{a}}_{\text{x}}$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{a}}_{\text{y}}$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{a}}_{\text{z}}$	0	0	1

Next, form $^N\vec{\boldsymbol{\omega}}^A$ (A's angular velocity in N) in terms of $\dot{\theta}$ and one of $\widehat{\mathbf{a}}_{\text{x}}$, $\widehat{\mathbf{a}}_{\text{y}}$, $\widehat{\mathbf{a}}_{\text{z}}$.

Result: $^N\vec{\boldsymbol{\omega}}^A = \dot{\theta}\, \widehat{\mathbf{a}}_{\text{z}}$

(b) The magnitude of $\widehat{\mathbf{a}}_{\text{x}}$ **does**/[✓ **does not**] change because $|\widehat{\mathbf{a}}_{\text{x}}| = 1$.
The direction of $\widehat{\mathbf{a}}_{\text{x}}$ [✓ **does**]/**does not** change in reference frame N as the baseball curves foul.
Note: If you have little baseball experience, consider the trajectory of a Frisbee that curves (does not fly straight).

(c) By **looking at the picture**, express $\vec{\mathbf{p}}$ (P's position from N_{o}) in terms of $\widehat{\mathbf{a}}_{\text{x}}$, $\widehat{\mathbf{a}}_{\text{y}}$, $\widehat{\mathbf{a}}_{\text{z}}$.
Next, use the rotation table to express $\vec{\mathbf{p}}$ in terms of $\widehat{\mathbf{n}}_{\text{x}}$, $\widehat{\mathbf{n}}_{\text{y}}$, $\widehat{\mathbf{n}}_{\text{z}}$.

Result: $\vec{\mathbf{p}} = r\,\widehat{\mathbf{a}}_{\text{x}} + z\,\widehat{\mathbf{a}}_{\text{z}} \qquad \vec{\mathbf{p}} = r\,\cos(\theta)\,\widehat{\mathbf{n}}_{\text{x}} + r\,\sin(\theta)\,\widehat{\mathbf{n}}_{\text{y}} + z\,\widehat{\mathbf{n}}_{\text{z}}$

These expressions show $\vec{\mathbf{p}}$ is a vector function of [✓ r]/θ/[✓ z] in $\boldsymbol{A}$,
whereas $\vec{\mathbf{p}}$ is a vector function of [✓ r]/[✓ θ]/[✓ z] in $\boldsymbol{N}$.
} Circle the correct variables.

(d) Alternatively, the ***Cartesian coordinates*** x, y, z locate P from N_{o} as $\vec{\mathbf{p}} = x\,\widehat{\mathbf{n}}_{\text{x}} + y\,\widehat{\mathbf{n}}_{\text{y}} + z\,\widehat{\mathbf{n}}_{\text{z}}$.
Express x, y, z in terms of r, θ, z. Then, express r and θ in terms of x and y.

Result: [Note: The `atan2` function is described in Section 1.4.5 and is undefined if $x = y = 0$.]

$$x = r\,\cos(\theta) \qquad y = r\,\sin(\theta) \qquad z = z \qquad r = {}^+\sqrt{x^2 + y^2} \qquad \theta = \texttt{atan2}(y,\, x) \neq \texttt{atan2}(x,\, y)$$

(e)
- r, θ, z define a **unique** location of P in N. [✓ **True**]/**False**.
- P's location in N defines a **unique** value for each of r, θ, z. **True**/[✓ **False**].

Hint: If P is at N_{o} (or directly above or below N_{o}) $\widehat{\mathbf{a}}_{\text{x}}$ is undefined so θ is also undefined.

(f) **Even planar analysis has issues:** An engineer uses r and θ to describe P's location in N when it is constrained to the horizontal plane perpendicular to $\widehat{\mathbf{n}}_{\text{z}}$. What location of P causes θ to be indeterminate? P may pass through $x = y = 0$ where θ is indeterminate/undefined.

5.26 Cylindrical coordinates and vector differentiation via definition (uses Hw 5.25, Section 7.1).

Using definition (7.1) of a vector derivative, form the time-derivative in $\boldsymbol{A}$ of $\vec{\mathbf{p}}$ in terms of $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$. Next, form the time-derivative in $\boldsymbol{N}$ of $\vec{\mathbf{p}}$ in terms of r, θ, z and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$. Then, factor on $\dot{r}$ and $r\,\dot{\theta}$ as shown below and use the rotation table to re-express your result in terms of $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$.

Result: (in terms of $r, \theta, \dot{r}, \dot{\theta}, \dot{z}$). Note: Hw 5.25 showed $\vec{\mathbf{p}} = r\,\widehat{\mathbf{a}}_x + z\,\widehat{\mathbf{a}}_z$ and $\vec{\mathbf{p}} = r\,\cos(\theta)\,\widehat{\mathbf{n}}_x + r\,\sin(\theta)\,\widehat{\mathbf{n}}_y + z\,\widehat{\mathbf{n}}_z$.

$$\frac{{}^{A}d\vec{\mathbf{p}}}{dt} = \dot{r}\,\widehat{\mathbf{a}}_x + \dot{z}\,\widehat{\mathbf{a}}_z$$

$$\begin{aligned}\frac{{}^{N}d\vec{\mathbf{p}}}{dt} &= [\dot{r}\,\cos(\theta) - r\,\sin(\theta)\,\dot{\theta}]\,\widehat{\mathbf{n}}_x + [\dot{r}\,\sin(\theta) + r\,\cos(\theta)\,\dot{\theta}]\,\widehat{\mathbf{n}}_y + \dot{z}\,\widehat{\mathbf{n}}_z \\ &= \dot{r}\,\underbrace{[\cos(\theta)\,\widehat{\mathbf{n}}_x + \sin(\theta)\,\widehat{\mathbf{n}}_y]}_{\widehat{\mathbf{a}}_x} + r\,\dot{\theta}\,\underbrace{[\text{-}\sin(\theta)\,\widehat{\mathbf{n}}_x + \cos(\theta)\,\widehat{\mathbf{n}}_y]}_{\widehat{\mathbf{a}}_y} + \dot{z}\,\widehat{\mathbf{n}}_z \\ &= \dot{r}\,\widehat{\mathbf{a}}_x + r\,\dot{\theta}\,\widehat{\mathbf{a}}_y + \dot{z}\,\widehat{\mathbf{a}}_z\end{aligned}$$

$$\frac{{}^{A}d\vec{\mathbf{p}}}{dt} \neq \frac{{}^{N}d\vec{\mathbf{p}}}{dt}$$ (fill in the blank with = or ≠).

5.27 Cylindrical coordinates and vector differentiation via angular velocity. (Section 7.1).

Referring to Hw 5.26, use the ***golden rule for vector differentiation*** (given below) and A's angular velocity in N (which is ${}^{N}\vec{\boldsymbol{\omega}}^{A} = \dot{\theta}\,\widehat{\mathbf{a}}_z$) to calculate the time-derivative in $\boldsymbol{N}$ of $\vec{\mathbf{p}}$.

Result:
Just calculate!

$$\frac{{}^{N}d\vec{\mathbf{p}}}{dt} = \underbrace{\frac{{}^{A}d\vec{\mathbf{p}}}{dt}}_{\dot{r}\,\widehat{\mathbf{a}}_x + \dot{z}\,\widehat{\mathbf{a}}_z} + \underbrace{{}^{N}\vec{\boldsymbol{\omega}}^{A} \times \vec{\mathbf{p}}}_{r\,\dot{\theta}\,\widehat{\mathbf{a}}_y}$$

Wow, the *golden rule for vector differentiation* is easier and more efficient than Hw 5.26!

5.28 ♣ Memorize the "golden rule for vector differentiation".

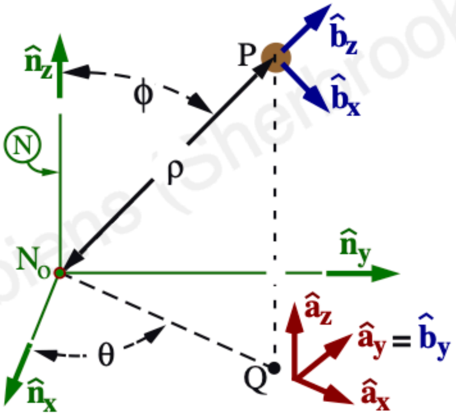
★ **Very important formula** ★

$$\boxed{\frac{{}^{N}d\vec{\mathbf{p}}}{dt} \underset{(7.7)}{=} \frac{{}^{A}d\vec{\mathbf{p}}}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{A} \times \vec{\mathbf{p}}}$$

Complete the equation that relates time-derivatives of **any vector** $\vec{\mathbf{p}}$ in **any frames** N and A via ${}^{N}\vec{\boldsymbol{\omega}}^{A}$ (A's angular velocity in N).

5.29 Spherical coordinates, position, and orientation. (Section 7.1).

The following figure shows a baseball P moving in a reference frame N. Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N as shown. P's location from point N_o (a point fixed in N) can be specified with time-dependent ***spherical coordinates***.



Right-handed sets of orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in reference frames A and B, respectively.

$\widehat{\mathbf{a}}_x$ points from N_o to point Q (point Q traces out P's projection onto the plane perpendicular to $\widehat{\mathbf{n}}_z$ and passing through N_o).

$\widehat{\mathbf{a}}_z$ points vertically-upward ($\widehat{\mathbf{a}}_z = \widehat{\mathbf{n}}_z$)

$\widehat{\mathbf{b}}_z$ points from N_o to P and $\widehat{\mathbf{b}}_y = \widehat{\mathbf{a}}_y$.

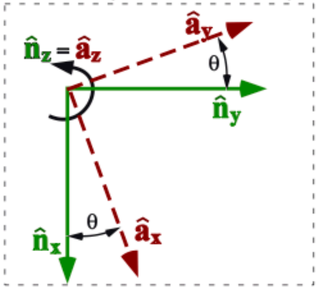
ρ	distance between N_o and P.	$0 \le \rho$
θ	angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{a}}_x$ with ${}^{+}\widehat{\mathbf{n}}_z$ sense.	$-\pi < \theta \le \pi$
ϕ	angle **between** $\widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_z$.	$0 \le \phi \le \pi$

(a)
- The magnitude of $\widehat{\mathbf{a}}_z$ **changes/[✓ stays constant]** with time.
- The direction of $\widehat{\mathbf{a}}_z$ in N **changes/[✓ stays constant]** with time.
- The direction of $\widehat{\mathbf{b}}_z$ in N **[✓ changes]/stays constant** with time.

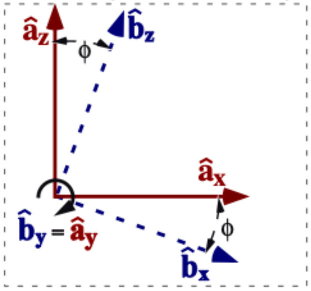
(b) **Draw** $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ in a geometrically helpful way to form the ${}^{a}R^{n}$ rotation table. Similarly, **draw** $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and form the ${}^{b}R^{a}$ rotation table. Next, use matrix multiplication to form the ${}^{b}R^{n}$ rotation table. Attempt to draw a 3D picture to form the ${}^{b}R^{n}$ rotation table directly from the picture and geometry. It is easier to use matrix multiplication to form ${}^{b}R^{n}$ than to draw a 3D picture and use geometry to form ${}^{b}R^{n}$. **[✓ True]/False** (circle one).

Result:

Draw $\widehat{\mathbf{a}}_x, \widehat{\mathbf{a}}_y, \widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$



Draw $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{a}}_x, \widehat{\mathbf{a}}_y, \widehat{\mathbf{a}}_z$



Attempt $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$

${}^aR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{a}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{a}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{a}}_z$	0	0	1

${}^bR^a$	$\widehat{\mathbf{a}}_x$	$\widehat{\mathbf{a}}_y$	$\widehat{\mathbf{a}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\phi)$	0	$-\sin(\phi)$
$\widehat{\mathbf{b}}_y$	1	1	0
$\widehat{\mathbf{b}}_z$	$\sin(\phi)$	0	$\cos(\phi)$

${}^bR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\phi)\,\cos(\theta)$	$\cos(\phi)\,\sin(\theta)$	$-\sin(\phi)$
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	$\sin(\phi)\,\cos(\theta)$	$\sin(\phi)\,\sin(\theta)$	$\cos(\phi)$

(c) By **inspection**, express $\vec{\mathbf{p}}$ (P's position from N_o) in terms of $\widehat{\mathbf{b}}_x, \widehat{\mathbf{b}}_y, \widehat{\mathbf{b}}_z$. Then use the rotation table to express $\vec{\mathbf{p}}$ in terms of $\widehat{\mathbf{n}}_x, \widehat{\mathbf{n}}_y, \widehat{\mathbf{n}}_z$. Express results in spherical coordinates ρ, θ, ϕ.

Result: $\vec{\mathbf{p}} = \rho\,\widehat{\mathbf{b}}_z$ $\quad\quad$ $\vec{\mathbf{p}} = \rho\,\sin(\phi)\,\cos(\theta)\,\widehat{\mathbf{n}}_x + \rho\,\sin(\phi)\,\sin(\theta)\,\widehat{\mathbf{n}}_y + \rho\,\cos(\phi)\,\widehat{\mathbf{n}}_z$

These expressions show $\vec{\mathbf{p}}$ is a vector function of $\boxed{\rho}/\theta/\phi$ in $\boldsymbol{B}$,
whereas $\vec{\mathbf{p}}$ is a vector function of $\boxed{\rho}/\boxed{\theta}/\boxed{\phi}$ in $\boldsymbol{N}$.
Circle the correct variables.

(d) Alternatively, the ***Cartesian coordinates*** x, y, z locate P from N_o as $\vec{\mathbf{p}} = x\,\widehat{\mathbf{n}}_x + y\,\widehat{\mathbf{n}}_y + z\,\widehat{\mathbf{n}}_z$. Express x, y, and z in terms of ρ, θ, and ϕ.

Result: $x = \rho\,\sin(\phi)\,\cos(\theta)$ $\quad\quad$ $y = \rho\,\sin(\phi)\,\sin(\theta)$ $\quad\quad$ $z = \rho\,\cos(\phi)$

(e) Express ρ, θ, ϕ in terms of x, y, z and calculate ρ, θ, ϕ when $x = 0,\ y = 0,\ z = 1$.

Note: The `atan2` function is described in Section 1.4.5 and is undefined if $x = y = 0$.

Result:

$\rho = {}^{+}\sqrt{x^2 + y^2 + z^2}$ $\quad\quad$ $\theta = \mathtt{atan2}(y,\, x)$ $\quad\quad$ $\phi = \mathrm{acos}(\frac{z}{\sqrt{x^2+y^2+z^2}})$

$\rho = 1$ $\quad\quad$ $\theta = ??$ $\quad\quad$ $\phi = 0°$

(f) Consider a baseball P (modeled as a particle) that is tied by a string to N_o. An engineer uses just two variables (angles θ and ϕ) to describe P's motion in N. Assuming $\widehat{\mathbf{n}}_z$ is ***vertically-upward*** (opposite gravity), what is P's location when its motion in N is damped out?

Result: P will come to rest directly below point N_o, where θ is indeterminate/undefined.

(g) Now rotate the page so $\widehat{\mathbf{n}}_z$ is ***horizontally-right*** (so gravity is in the $\widehat{\mathbf{n}}_y$ direction). What locations of P cause θ to be indeterminate?

Result: If P passes horizontally-right or left of point N_o, θ is indeterminate/undefined.

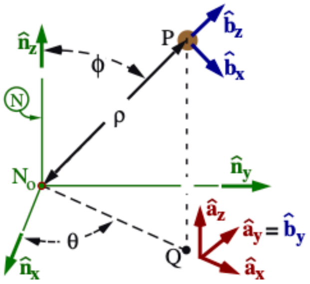
Note: Although P's motion in N has **2** degrees of freedom, its motion can be described without "singularities" by using the **3** Cartesian Coordinates x, y, z and imposing the **1** *configuration constraint* $x^2 + y^2 + z^2 = 25$. The idea of using additional coordinates to avoid singularities is why a quaternion (**4** Euler parameters) is used to describe the general **3D** orientation of a rigid body in space.

Spherical coordinates help predict river flow and bank erosion on Earth

5.30 Spherical coordinates and vector differentiation via definition. (Section 7.1).

Referring to Hw 5.29, use vector derivative definition (7.1) to form the time-derivative in N of $\vec{\mathbf{p}}$ in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$. Then, use the rotation table to re-express your result in terms of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$.

Result: (in terms of ρ, θ, ϕ, $\dot{\rho}$, $\dot{\theta}$, $\dot{\phi}$).



$$\frac{{}^{N}d\vec{\mathbf{p}}}{dt} = \left[\sin(\phi)\,\cos(\theta)\,\dot{\rho} + \rho\,\cos(\phi)\,\cos(\theta)\,\dot{\phi} - \rho\,\sin(\phi)\,\sin(\theta)\,\dot{\theta}\right]\hat{\mathbf{n}}_x$$

$$+ \left[\sin(\phi)\,\sin(\theta)\,\dot{\rho} + \rho\,\sin(\phi)\,\cos(\theta)\,\dot{\theta} + \rho\,\sin(\theta)\,\cos(\phi)\,\dot{\phi}\right]\hat{\mathbf{n}}_y$$

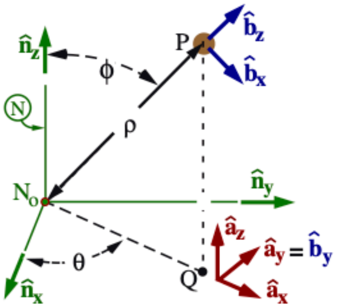
$$+ \left[\cos(\phi)\,\dot{\rho} - \rho\,\sin(\phi)\,\dot{\phi}\right]\hat{\mathbf{n}}_z$$

$$\frac{{}^{N}d\vec{\mathbf{p}}}{dt} = \rho\,\dot{\phi}\,\hat{\mathbf{b}}_x + \rho\,\sin(\phi)\,\dot{\theta}\,\hat{\mathbf{b}}_y + \dot{\rho}\,\hat{\mathbf{b}}_z$$

Note: Attempt to form this 2^{nd} expression by doing **laborious** trigonometric simplifications until it is clear how laborious this is.

The expression for $\frac{{}^{N}d\vec{\mathbf{p}}}{dt}$ is shorter when expressed in terms of $\boxed{(\hat{\mathbf{b}}_x,\ \hat{\mathbf{b}}_y,\ \hat{\mathbf{b}}_z)}$ / $(\hat{\mathbf{n}}_x,\ \hat{\mathbf{n}}_y,\ \hat{\mathbf{n}}_z)$.

5.31 Spherical coordinates and vector differentiation via angular velocity. (Section 7.1).



Inspect the figure to determine P's position from N_o in terms of $\hat{\mathbf{b}}_z$. Calculate $\vec{\mathbf{p}}$'s time-derivative in frame B in terms of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$.

Result: $\vec{\mathbf{p}} = \rho\,\hat{\mathbf{b}}_z$ $\qquad \frac{{}^{B}d\vec{\mathbf{p}}}{dt} = \dot{\rho}\,\hat{\mathbf{b}}_z$

Given: A's angular velocity in N, B's angular velocity in A, and ${}^{N}\vec{\boldsymbol{\omega}}^{B}$:

$${}^{N}\vec{\boldsymbol{\omega}}^{A} = \dot{\theta}\,\hat{\mathbf{a}}_z \qquad {}^{A}\vec{\boldsymbol{\omega}}^{B} = \dot{\phi}\,\hat{\mathbf{b}}_y \qquad {}^{N}\vec{\boldsymbol{\omega}}^{B} = {}^{N}\vec{\boldsymbol{\omega}}^{A} + {}^{A}\vec{\boldsymbol{\omega}}^{B} = \dot{\theta}\,\hat{\mathbf{a}}_z + \dot{\phi}\,\hat{\mathbf{b}}_y$$

Referring to Hw 5.30, use the ***golden rule for vector differentiation*** (given below) and B's angular velocity in N (which is ${}^{N}\vec{\boldsymbol{\omega}}^{B} = \dot{\theta}\,\hat{\mathbf{a}}_z + \dot{\phi}\,\hat{\mathbf{b}}_y$) to calculate the time-derivative in $\boldsymbol{N}$ of $\vec{\mathbf{p}}$.

Result:
Just calculate!

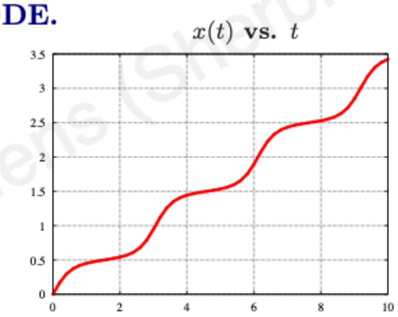
$$\frac{{}^{N}d\vec{\mathbf{p}}}{dt} = \underbrace{\frac{{}^{B}d\vec{\mathbf{p}}}{dt}}_{\dot{\rho}\,\hat{\mathbf{b}}_z} + \underbrace{{}^{N}\vec{\boldsymbol{\omega}}^{B} \times \vec{\mathbf{p}}}_{\rho\,\dot{\phi}\,\hat{\mathbf{b}}_x \,+\, \rho\,\sin(\phi)\,\dot{\theta}\,\hat{\mathbf{b}}_y}$$

Wow, the *golden rule for vector differentiation* is easier and more efficient than Hw 5.30!

5.32 †Optional: Continuous numerical solution of a nonlinear ODE.

Plot the continuous solution $x(t)$ to the following ordinary differential equation for $0 \le t \le 10$ with data every 0.2 sec. Use an initial value $x(0) = 0$ and use the initial value of $\dot{x}$ that is closest to 1.

$$\sin(\dot{x}) + 4\,\dot{x}^2 - 1.9\,\cos(2\,\pi\,x) - 2 = 0$$


$x(t)$ **vs.** t

Hint: A "clever" way to solve this **nonlinear** ODE for $x(t)$ is

- Use the given equation and initial value $x(0) = 0$ to solve for $\dot{x}$ at $t = 0$.
  For example, the technique in Section 1.12 finds $\dot{x}(t{=}0) \approx 0.8841161$ when $x(t{=}0) = 0$.
- Differentiate the 1^{st}-order ODE (which is **nonlinear** in $\dot{x}$) to form a 2^{nd}-order ODE (which is **linear** in $\ddot{x}$). Then, solve the 2^{nd}-order ODE for $\ddot{x}$.

$$\cos(\dot{x})\,\ddot{x} + 8\,\dot{x}\,\ddot{x} + 3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x} = 0 \qquad \Rightarrow \qquad \ddot{x} = \frac{-3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x}}{\cos(\dot{x}) + 8\,\dot{x}}$$

- Numerically integrate the 2^{nd}-order ODE with the initial values of $x(0)$ and $\dot{x}(0)$
